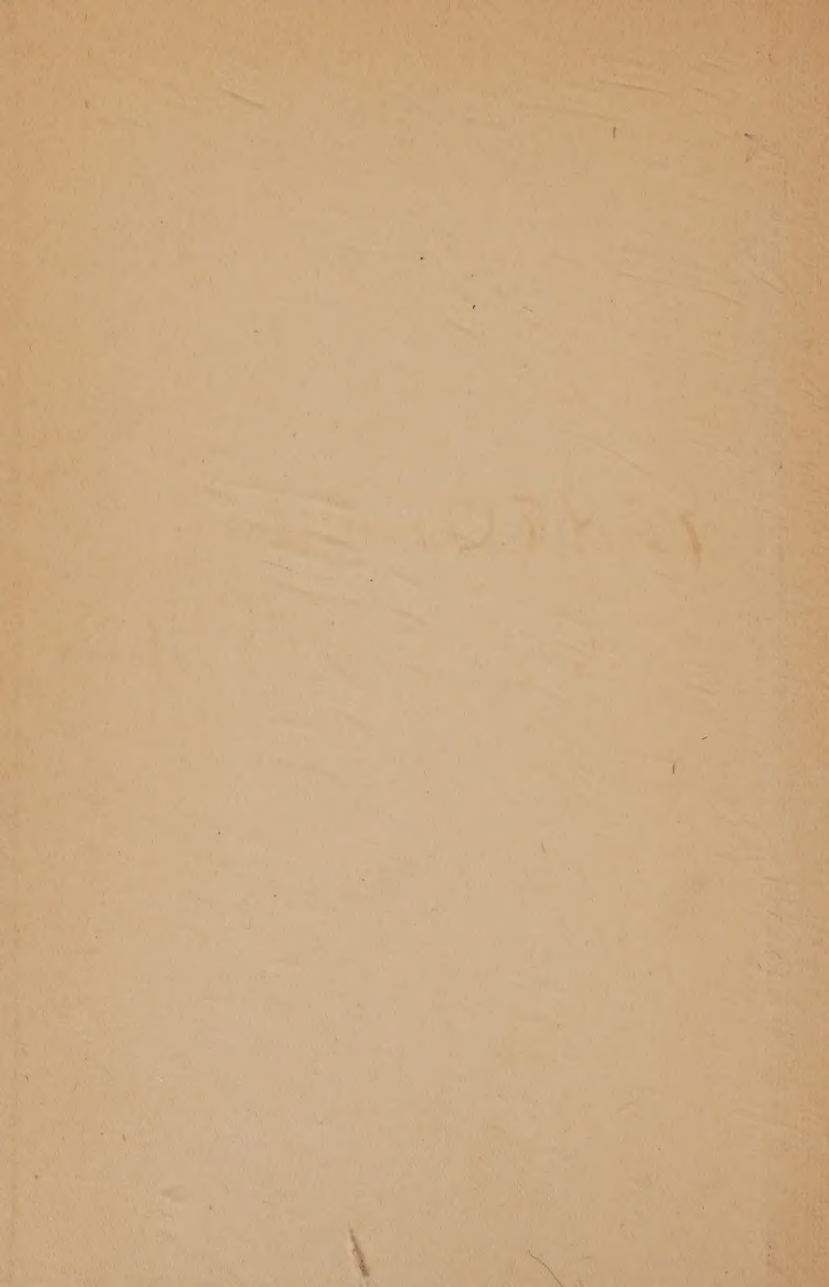




Albert F. Chamberlain

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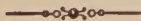
ELEMENTS OF ALGEBRA

A COURSE FOR GRAMMAR SCHOOLS AND BEGINNERS
IN PUBLIC AND PRIVATE SCHOOLS

BY

WILLIAM J. MILNE, PH.D., LL.D.

PRESIDENT OF NEW YORK STATE NORMAL COLLEGE, ALBANY, N.Y



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MILNE'S MATHEMATICS

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MILNE'S PLANE GEOMETRY—SEPARATE

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Elmts. Alg.

E-P

PREFACE.

ALGEBRA has not always proved to be an interesting subject to the younger classes in our secondary or lower schools; indeed, in very many instances it has been greatly disliked by the students in such institutions. Two causes, chiefly, have conspired to produce this unfortunate condition of affairs, — one, the unattractive and uninteresting method of presenting the subject; the other, the difficulty of the examples and the complexity of the problems presented to the pupils for solution.

It is believed that this text-book presents the elementary facts of the science in such a manner that a deep interest will be awakened in algebraic processes, and that the examples which the student is required to solve are quite within the scope of his ability to master.

The author has in several instances departed from the order of classification commonly followed in text-books on algebra, because he has preferred to arouse an interest in the subject rather than to follow an order which is scientific, but which does not introduce the student to the attractive features of the study until he has mastered all the processes employed with most forms of algebraic quantities. And yet in no instance have erroneous mathematical ideas

been taught, nor has correct reasoning ever been sacrificed for the purpose of exciting such interest.

The ideas of number which the pupil has gained in arithmetic have been associated with those involved in algebra in such a way that no difficulty may be experienced in passing from reasoning upon definite numbers to reasoning upon general numbers.

The treatment of equations is introduced at the beginning, and it is presented throughout the book, wherever it is possible to do so advantageously, because, since elementary algebra treats of almost nothing except the equation and general numbers, the student should be led to a comprehension of the simpler forms of the equation as soon as possible.

The method of presentation exemplified in the other books of the series has been followed here, because it is recognized as pedagogically correct and because it has met with general approbation.

The work is designed to present the merest elements of the science, and yet it is believed that the method of presentation, the illustration and application of mathematical principles, and the knowledge gained from the solution of problems, will familiarize the student with the fundamental principles of the science to such a degree that easy and rapid progress in the more abstract phases of the subject will be secured whenever he pursues the subject farther.

WILLIAM J. MILNE.

STATE NORMAL COLLEGE,
Albany, N.Y.

CONTENTS.



	PAGE
ALGEBRAIC PROCESSES	7
ALGEBRAIC EXPRESSIONS	19
Terms in Algebraic Expressions	23
Positive and Negative Quantities	25
ADDITION	26
SUBTRACTION	32
Transposition in Equations	41
Equations and Problems	43
MULTIPLICATION	47
Special Cases in Multiplication	54
Simultaneous Equations	58
DIVISION	63
GENERAL REVIEW EXERCISES	70
FACTORING	73
Equations solved by Factoring	80
COMMON DIVISORS OR FACTORS	85
COMMON MULTIPLES	88
FRACTIONS	91
Reduction of Fractions	92
Clearing Equations of Fractions	101
Addition and Subtraction of Fractions	107
Multiplication of Fractions	110
Division of Fractions	114

	PAGE
EQUATIONS	116
Review of Equations	120
INVOLUTION	131
EVOLUTION	134
QUADRATIC EQUATIONS	150
Pure Quadratic Equations	150
Affected Quadratic Equations	152
Simultaneous Quadratic Equations	158
GENERAL REVIEW	162
QUESTIONS FOR REVIEW	177
ANSWERS	183

ELEMENTS OF ALGEBRA.



ALGEBRAIC PROCESSES.

1. PROBLEM. A farmer had 444 sheep in two fields, one of which contained three times as many sheep as the other. How many sheep had he in each field?

ARITHMETICAL SOLUTION.

A certain number = the number in one field.

Then 3 times that number = the number in the other field.

And 4 times that number = the number in both fields.

Therefore 4 times that number = 444.

The number = 111, the number in one field.

3 times 111 = 333, the number in the other field.

The solution given above may be abbreviated by using the letter n for the expressions *a certain number* and *that number*. In algebra, however, it is customary to represent a number whose value is to be found by x or by some other one of the last letters of the alphabet.

ALGEBRAIC SOLUTION.

Let x = the number in one field.

Then $3x$ = the number in the other field.

And $4x$ = the number in both fields.

Therefore $4x = 444$.

$x = 111$, the number in one field.

$3x = 333$, the number in the other field.

2. The student will observe that algebra is very much like arithmetic, but that it differs from arithmetic in often using letters to represent numbers.

Letters used to represent numbers are usually called *quantities*, though it is also proper to call them numbers.

Thus, x in the preceding algebraic solution is called a quantity.

3. Letters used to represent numbers or quantities whose values are to be found are called **Unknown Numbers or Quantities**.

The *last letters* of the alphabet, as x , y , z , etc., are commonly employed to represent *unknown numbers or quantities*.

Thus, x in the solution already given is called an unknown quantity.

4. An expression of equality between two numbers or quantities is called an **Equation**.

Thus, $5 + 8 = 13$, and $3x = 15$ are called equations.

5. The sign $=$ is used to indicate equality.

6. The sign $\therefore$ is used, in writing solutions of problems, instead of the word *therefore* or *hence*. It is called the sign of **Deduction**.

PROBLEMS.

7. 1. John and James together had 40 marbles, but John had three times as many as James. How many had each?

2. Two boys together dug 60 bushels of potatoes. If one dug twice as many bushels as the other, how many did each dig?

3. If A and B together have \$3000, and A has twice as much as B, how much has each of them?

4. A number added to five times itself equals 30. What is the number?

5. If a man is three times as old as his son, and the sum of their ages is 40 years, how old is the father? How old is the son?

6. Divide the number 75 into two parts such that one part may be four times the other.

7. A man bought a horse and carriage for \$600, and the carriage cost him twice as much as the horse. What did each cost him?

8. A certain orchard has 700 trees in it. There are twice as many cherry trees as pear trees, and twice as many apple trees as cherry trees. How many are there of each kind?

9. A, B, and C were talking of their ages. A said, "I am twice as old as B"; B said, "I am twice as old as C." The sum of their ages was 140 years. How old was each?

SUGGESTION. Discover from the problem the number which, if known, will enable you to find all the required numbers, and let x stand for that number. In this problem x will stand for C's age.

10. The greater of two numbers is six times the less, and their sum is 49. What are the numbers?

11. After taking five times a number from fifteen times the same number, and adding to the remainder six times the number, the result was 8 more than 72. What was the number?

12. Mary has five times as many books as Hannah, Hannah has two times as many as Jane, and together they have 39 books. How many has each?

13. I paid five times as much for pens as for ink, and three times as much for paper as for both of these. How much did I pay for each, if they all cost me 96 cents?

14. A man paid twice as much for a pair of trousers and seven times as much for an overcoat as for a hat. How much did each cost him, if they all cost \$60?

15. A man divided \$50 among four boys. To the second he gave twice, to the third three times, and to the fourth four times, as much as to the first. How much did he give to each?

16. A farmer has four flocks of sheep. The second is twice as large as the first, the third three times as large, and the fourth as large as the first and third together. If he has 200 sheep in all, how many are there in each flock?

17. If one number is six times another number, and a third number is 10 times that number, and the sum of the three is 119, what are the numbers?

18. A man failed in business, owing A ten times as much as B, C four times as much as B, and D eight times as much as the difference between what he owed A and C. The entire amount he owed these men was \$6300. How much did he owe each?

19. If a gold watch is worth ten times as much as a silver watch, and both together are worth \$132, how much is each watch worth?

20. The whole number of votes cast for A and B at a certain election was 450, but four times as many votes were cast for A as for B. How many were cast for each?

21. A man paid \$175 for a horse, a cow, and a wagon. He paid twice as much for the horse as for the wagon, and

twice as much for the wagon as for the cow. How much did he pay for each?

22. A, B, and C form a partnership with a capital of \$14,000. B and C each furnish three times as much as A. How much capital does each furnish?

23. A and B performed a certain piece of work for which B received three times as much as A, and A received \$12 less than B. How much did each receive?

SOLUTION. Let x = the amount A received.

Then $3x$ = the amount B received.

$2x$ = the amount B received more than A.

$\therefore 2x = \$12$.

$x = \$6$, the amount A received.

$3x = \$18$, the amount B received.

24. A farmer raised five times as many acres of wheat as of oats, and five times as many acres of oats as of potatoes. If 186 acres were cultivated, how many acres were there of each?

25. A man's capital doubled for three successive years. If he then had \$9600, how much had he at first?

SUGGESTION. Let x = his original capital.

Then $2x$ = his capital at end of first year.

26. Sarah had seven times as many chickens as ducks, and twice as many ducks as hens. If there were in all 119 fowls, how many were there of each kind?

27. A certain number can be separated into two factors, one of which is three times the other, and whose sum is 36. What are the two factors?

28. One cask holds four times as much as another, and the larger contains 27 gallons more than the smaller. How many gallons are there in each cask?

29. The population of one town is eight times the population of another, and the larger one contains 4900 more inhabitants than the smaller. What is the population of each town?

30. The difference between a certain number and that number multiplied by 18, is 119. What is the number?

31. A, B, and C work the same number of days. A receives 75 cents per day, B receives 90 cents, and C receives 95 cents. All together receive \$13. How many days does each work?

32. A father's age is five times the age of his son, and the difference between their ages is 44 years. What is the age of each?

33. What number added to $\frac{1}{2}$ of itself equals 12?

SOLUTION. Let x = the number.

Then $x + \frac{1}{2}x = 12$

Or $\frac{3}{2}x = 12$

$\therefore \frac{1}{2}x = 4$

And $x = 8$

Hence the number is 8.

34. If a certain number is added to $\frac{1}{3}$ of itself, the sum is 20. What is the number?

35. If to six times a certain number $\frac{1}{3}$ of itself is added, the sum is 19. What is the number?

36. A farmer had $\frac{3}{7}$ as many horses as cows, and altogether he had 10 cows and horses. How many of each had he?

37. One of my pear trees bore $\frac{2}{7}$ as many bushels of pears as the other, and both together bore 18 bushels. How many did each tree bear?

38. There is a certain number, $\frac{1}{4}$ of which exceeds $\frac{1}{6}$ of it by 2. What is the number?

39. One half of a certain number exceeds $\frac{1}{3}$ of it by 3. What is the number?

40. Divide 45 into two parts such that one may be $\frac{2}{3}$ of the other.

41. One half and one third of a certain number added together make 10. What is the number?

42. Three fourths of a number added to two fifths of it makes 23. What is the number?

43. If Charles had five times as many marbles as he now has, he would have as many as John and William together, the former of whom has 30, and the latter 15. How many marbles has Charles?

44. If a fourth of a number is subtracted from $\frac{1}{2}$ the number, the result is 3. What is the number?

45. A man had 260 sheep in three fields. In the second he had twice as many as in the first, and in the third $\frac{1}{4}$ of the number in the first. How many had he in each field?

46. The less of two numbers is $\frac{1}{3}$ of the greater, and their sum is 14 less than 50. What are the numbers?

47. Divide 120 into three such parts that the first part is $\frac{1}{2}$ of the second, and $\frac{1}{3}$ of the third.

SUGGESTION. Let $x =$ the first part.

Then $2x =$ the second part.

48. A, B, and C together have \$900. A has twice as much as B, and C has twice as much as A and B together. How much has each?

49. What number is that to which if you add its half and take away its third, the remainder will be 147?

50. James has $\frac{1}{3}$ as many marbles as Henry, and Henry has 16 more than James. How many marbles has each?

51. Julia has $\frac{1}{3}$ as much money as May; Hattie has $\frac{1}{2}$ as much as Julia; all together have 18 cents. How many cents has each?

52. A certain number plus $\frac{1}{3}$ of itself, plus $\frac{1}{5}$ of itself equals 46. What is the number?

53. In Mr. Green's library there were 1900 volumes. The number of books of fiction was $\frac{2}{5}$ the number on science; the number on science was twice the number on history. How many volumes were there of each?

54. What number increased by $\frac{9}{10}$ of itself equals 133?

55. B has $\frac{3}{5}$ as much money as A; C has $\frac{1}{3}$ as much as B; altogether they have \$ 27. How much has each?

56. Two fifths of a number plus $\frac{1}{2}$ of the number equals 36. What is the number?

57. George sold a certain number of oranges at 2 cents apiece; Walter sold $\frac{1}{3}$ as many at 4 cents apiece. If both together received 90 cents, how many oranges did each sell?

SOLUTION. Let x = the number George sold.

Then $\frac{1}{3}x$ = the number Walter sold.

$2x$ = the number of cents George received.

$\frac{4}{3}x$ = the number of cents Walter received.

$\frac{4}{3}x + 2x$ = the number of cents both received.

$\therefore \frac{10}{3}x = 90.$

$\frac{1}{3}x = 9.$

$x = 27$, the number of oranges George sold.

$\frac{1}{3}x = 9$, the number of oranges Walter sold.

58. Robert bought a certain number of apples, and his mother gave him $\frac{4}{5}$ as many as he bought. He then had 18. How many apples did he buy?

59. A man gave \$50 to three poor families. To the first he gave $\frac{1}{2}$ as much as to the second, to the third seven times as much as to the first. How much did he give to each?

60. Three newsboys sold 100 papers. The first and the second sold the same number, while the third sold $\frac{2}{3}$ as many as the other two. How many did each sell?

61. A and B start in business with the same amount of money. A gains $\frac{1}{2}$ as much as he had, while B loses $\frac{1}{5}$ of his capital; they then have together \$1150. What was the original capital of each?

62. What number increased by $\frac{1}{9}$ of $\frac{3}{4}$ of itself equals 52?

63. The length of a field is $1\frac{1}{2}$ times its breadth, and the entire distance around the field is 60 rods. What are the length and the breadth of the field?

SOLUTION. Let x = the number of rods in the breadth.

Then $1\frac{1}{2}x$ = the number of rods in the length.

And $2\frac{1}{2}x$ = half the distance around the field.

$\therefore 5x = 60$ rods, the distance around the field.

$x = 12$ rods, the breadth of the field.

$1\frac{1}{2}x = 18$ rods, the length of the field.

64. What is the number whose double and half added together give 35?

65. The sum of three numbers is 34. The first multiplied by 4 produces the second, and $\frac{1}{2}$ of the first taken from the second gives the third. What are the numbers?

66. Find two factors of 100, one of which is four times the other, and whose sum is $\frac{1}{4}$ of 100.

67. A father gave to his first son three times as much as to the second, to the third $\frac{1}{2}$ as much as to the first and second; in all he gave \$ 600. How much did he give to each?

68. Four men are taxed in the aggregate \$ 500 as follows: for every dollar that the first pays the second pays 1 dollar, the third pays $\frac{1}{2}$ of a dollar, and the fourth pays $1\frac{1}{2}$ dollars. How much is each man's tax?

69. Emma sold 10 quarts of berries at a certain price per quart. Anna sold 14 quarts at twice that price per quart. Together they received \$ 1.90. What price per quart did each receive?

70. Two boys walked toward each other from two towns 18 miles apart. If one boy walked $\frac{4}{5}$ as far as the other, how many miles did each walk?

71. What number is that whose fourth part exceeds its fifth part by 3?

72. A man paid a certain sum for a house and lot, and sold them at a gain equal to $\frac{1}{2}$ the cost. If the selling price was \$ 3000, what was the cost?

SOLUTION. Let x = the cost of the house and lot.

Then $\frac{1}{2}x$ = the gain.

And $1\frac{1}{2}x$ = the selling price.

$$\therefore \frac{3}{2}x = 3000.$$

$$\frac{1}{2}x = 1000.$$

$x = 2000$, the cost of the house and lot.

73. A man paid a debt of \$ 7500 in 4 months, paying each month twice as much as the month before. How much did he pay the first month?

74. A man traveled 324 miles; he went three times as far by steamboat as by stage, and eight times as far by railroad as by steamboat. How many miles did he travel by each conveyance?

75. Harry solved 8 problems more than Edward, and Edward solved $\frac{1}{3}$ as many as Harry. How many did each solve?

76. A man paid \$150 for a wagon, which was 50 per cent or one half more than the original cost. What was the original cost of the wagon?

77. A man bought a lot for a certain sum; he built upon it a house costing twice as much as the lot; he furnished the house at an expense equal to $\frac{1}{2}$ the price of the lot. The man, who had only \$6000 in cash, found that he could pay for all his expenditures except furnishing the house. What was the amount of each item?

78. In an orchard there are 210 trees, arranged in rows. There are 5 rows with a certain number of trees in a row, and 8 rows with twice that number of trees in a row. How many trees are there in the different rows?

79. A man earned daily for 4 days three times as much as he paid for his board. After paying his board for 4 days, he had \$8 left. How much did he receive per day, and how much did he pay for his board?

80. A man on being asked how old he was said that $\frac{1}{10}$ of his age added to twice his age made 84 years. How old was he?

81. A man borrowed as much money as he had, and then spent $\frac{1}{3}$ of the whole sum. If he had \$4 left, how much had he at first?

82. A man had twice as many 5-cent pieces as dimes, and his money amounted to 200 cents. How many pieces had he of each kind?

SOLUTION. Let x = the number of dimes.

Then $2x$ = the number of 5-cent pieces.

And $10x$ = the number of cents in the dime pieces.

$10x$ = the number of cents in 5-cent pieces.

$$\therefore 20x = 200.$$

$x = 10$, the number of dimes.

$2x = 20$, the number of 5-cent pieces.

83. A sum of money amounting to \$5 is composed of half and quarter dollars. The number of half dollars is double the number of quarter dollars. How many pieces are there of each kind?

84. A man worked 10 days for a certain sum per day; his wife also worked at the same rate, and his son at $\frac{1}{3}$ the rate of his father. They received in all \$27.50. What were the daily wages of each?

85. The length of a rectangular field is twice the breadth, and the distance around the field is 90 rods. What are the length and the breadth of the field?

86. A tree 45 feet high was broken off so that the part left standing was twice the part broken off. What was the length of each part?

87. Divide \$800 among three men so that the first and third shall together have three times as much as the second, and the first shall have double what the third has.

88. The income from a very successful business quadrupled every year for three years. If the entire income for the three years was \$42000, what was the income for each year?

ALGEBRAIC EXPRESSIONS.

8. Quantities expressed by algebraic symbols are called **Algebraic Expressions**.

Thus, $a + b$ and $2a - 3bc$ are algebraic expressions.

9. The signs employed in arithmetic are generally used for the same purposes in algebra.

What do the following expressions indicate?

1. $2 + 5$; $17 - 6$; $16 \div 4$; 15×3 .
2. $a + b$; $c - d$; $c \div d$; $d \times a$.
3. $a + b + c$; $a - b + c$; $a - b - c$; $a \times b \times c$.
4. $a - b - c + d$; $a + b - d - e$; $a \times b \times c \times d$.

Multiplication is also indicated by writing letters, or a letter and a figure, side by side without any sign between them, or with a dot between them. Thus, $a \times b \times c$ may be written abc or $a \cdot b \cdot c$.

What do the following expressions indicate?

5. $4ab$; $5ab$; $3a + 2b$; $4x + 3y$.
6. $4xyz$; $6abc$; $6ax + 3ay$; $5cd - 3d$.
7. $3xy - 4yz + 2z$; $4xyz - 2axy - 3by$.

10. The quantities which, when multiplied together, form a product are called the **Factors** of the product.

11. The product arising from using a quantity a certain number of times as a factor is called a **Power** of that quantity.

Thus, 4 is a power of 2; 64 is a power of 4 and also of 8.

A power is indicated by a small figure or letter, called an **Exponent**, written a little above and at the right of the quantity, showing the number of times the quantity is to be used as a factor.

Thus, a^5 shows that a is to be used as a factor *five* times, or that it is equal to $a \times a \times a \times a \times a$.

Powers are named from the number of times a quantity is used as a factor.

Thus, a^7 is read the *seventh power* of a or a *seventh power*.

The *second* power is also called the *square*, and the *third* power the *cube* of a quantity.

12. Read the following expressions and state what they indicate.

1. a^8 ; x^6 ; x^9 ; x^2y^2 ; x^3y^3 ; x^2y^3 ; x^5y^4 .

2. $a^2 + b^2$; $a^3 + b^3$; $a^5 + b^5$; $a^8 - b^8$.

3. $a^2b + ab^2$; $a^3x - b^3y^2$; $a^2x^3 + bx^3$; $a^4x^2y - b^5x^3y^2$.

4. $a^2 + b^2 - c^2$; $a^2x^2 + b^2y^2 + c^2z^2$; $axy^3 + by^2z^2 - y^3$.

13. A figure or letter placed before a quantity to show how many times it is taken additively is called a **Coefficient**.

Thus, in the expression $5x$, 5 indicates that x is to be taken five times, or that it is equal to $x + x + x + x + x$.

In the expression $4bc$, 4 may be regarded as the coefficient of bc , or $4b$ may be regarded as the coefficient of c .

When no coefficient is written, it is manifestly 1.

14. What do the following expressions indicate?

1. $5a^2x^3z^5$.

INTERPRETATION. The expression indicates five times the product of a used twice as a factor, multiplied by x used three times as a factor, multiplied by z used five times as a factor.

It is usually read *five a square, x cube, z fifth power*.

2. $3x^2y^2$.

5. $2x^2 + y^2$.

8. $3x^2 + 2y - 3z$.

3. $4x^3y^3$.

6. $3x - 4y^2$.

9. $2a^2x^2 - 3xy + 2z^2$.

4. $5axy^4$.

7. $4xy - 6y^3$.

10. $5a^2y + 3x^2y^2 + 5cz^2$.

15. When several quantities are inclosed in parentheses, (), they are to be treated as a single quantity.

Thus, $2 \times (x + y)$ indicates two times the sum of x and y ; $3(a + b - c)$ indicates three times the remainder when c has been subtracted from the sum of a and b .

16. 1. Interpret and read the expression $a \times (b + c)$ or $a(b + c)$.

INTERPRETATION. The expression means that the sum of b and c is to be multiplied by a . It is usually read a times the *quantity b plus c*.

Interpret and read the following :

- | | |
|-----------------------|---------------------------------|
| 2. $x(y + z)$. | 7. $a + 4d(ax - cy)$. |
| 3. $5(a - c)$. | 8. $3x + 4(2y - 3z)$. |
| 4. $3d(c - y)$. | 9. $2(3x - 5y) + (6x - 3y)$. |
| 5. $4x(c + 2d)$. | 10. $3a(x + c) - 5y(z + d)$. |
| 6. $3ac(2x - 3y^2)$. | 11. $5ax(a - b) - 3cy(c + d)$. |

Write the following in algebraic expressions :

1. The sum of five times a and three times the square of x .

SOLUTION. $5a + 3x^2$.

2. Three times b , diminished by 5 times a raised to the fourth power.

3. The product of a , b , and $a - c$.

4. Seven times the product of x and y , increased by three times the cube of z .

5. Three times x , diminished by five times the sum of a , b , and c .

6. Six times the square of m , increased by the product of m and n .

7. The product of a used five times as a factor, multiplied by the sum of b and c .

8. Twelve times the square of a , diminished by five times the cube of b .

9. Eight times the product of a and b , diminished by four times the fourth power of c , or c used four times as a factor.

10. Six times the product of n and the second power of a ; increase this by five times the product of a and the second power of n .

11. Four times the product of x by the second power of y , multiplied by the cube of z , diminished by a times the fourth power of c .

12. The fourth power of a plus the cube of b , plus three times the product of the square of a by the square of b , diminished by the cube of d .

17. When $a = 1$, $b = 2$, $c = 3$, $d = 4$, $e = 5$, find the numerical value of each of the following expressions by using the number for the letter which represents it.

Thus, $5a + 6b - 2c^2 + d = 5 + 12 - 18 + 4 = 3$.

- | | |
|----------------------------|---------------------------------------|
| 1. $3abcd$. | 13. $12bc + e(c - a)$. |
| 2. $5abcd^2$. | 14. $3c^2d - 5abc$. |
| 3. $2a^2b^2c^2$. | 15. $9a - 4b + 2cd$. |
| 4. $3abcde$. | 16. $3abd - 4e + cd$. |
| 5. $3a + 5b$. | 17. $a^2b^2c^2 - d^2 + e^2$. |
| 6. $4a + 3b$. | 18. $(a + b) + e(d - c)$. |
| 7. $2e - 3c + d$. | 19. $b(d - a + c) - bd$. |
| 8. $4ab^2cd^2$. | 20. $abc + ade - d(e - c)$. |
| 9. $10a^2b - 3ab^2$. | 21. $b^2c^2d^2 - ab + cd$. |
| 10. $3d - 2b^2 + e$. | 22. $b(d + a) + e(b + c)$. |
| 11. $5cd - 5bc - 5ab$. | 23. $c^2 + 4abc + d(e - d)$. |
| 12. $8a^2b^3 + c^2 - de$. | 24. $(a + b) + 3(c + d) + 2(e - d)$. |

TERMS IN ALGEBRAIC EXPRESSIONS.

18. An algebraic expression whose parts are not separated by the signs $+$ or $-$ is called a **Term**.

Thus, $5x$, $3by$, $4x^2y^2z^2$, $a^2b^2d^2e$ are terms.

In the expression $3x + 2ay + 3cd - z$ there are *four* terms; in the expression $xy^2 - x^2y^2$ there are but *two* terms.

19. A term which has the sign $+$ before it is called a **Positive Term**. When the first term of an expression is positive, the sign $+$ is usually omitted before it.

Thus in the expression $2a - 3b + 4c - 5d + 3e$ the first, third, and fifth terms are *positive*.

20. A term which has the sign $-$ before it is called a **Negative Term**.

Thus in the expression $2a - 3b + 4c - 5d - 3e$, the second, fourth, and fifth terms are *negative*.

21. Terms which contain the same letters with the same exponents are called **Similar Terms**.

The coefficients or signs need not be alike, however.

Thus, $5x^3$ and $-7x^3$ are similar terms; also $3(x+a)^5$ and $5(x+a)^5$.

Expressions like bx^3 and cx^3 may be considered similar terms by regarding b and c as coefficients.

22. Terms which contain different letters, or the same letters with different exponents, are called **Dissimilar Terms**.

Thus, $3xy$ and $3pq$ are dissimilar terms; also $5x^2y$ and $5xy^2$.

23. An algebraic expression consisting of but one term is called a **Monomial**.

Thus, ab , $3axyz$, $5x^2y^2z$ are monomials.

24. An algebraic expression consisting of more than one term is called a **Polynomial**.

Thus, $a + b + c$ and $3x + 4y$ are polynomials.

25. A polynomial of two terms is called a **Binomial**.

Thus, $3x + 2y$, $a - b$, and $4ab - 3cd$ are binomials.

26. A polynomial of three terms is called a **Trinomial**.

Thus, $a + b + c$ and $2x - 2y + 5z$ are trinomials.

27. Select from the following :

1st, the positive terms.

5th, the monomials.

2d, the negative terms.

6th, the binomials.

3d, the similar terms.

7th, the trinomials.

4th, the dissimilar terms.

8th, the polynomials.

1. $2ax$, $3x^2y$, $2a + 3b$, $6x^2y$, $5ax^2$, $3x + 2b$.

2. $3x^2 + 2y^2$, $5a - 3b + c$, $-4ax$, $6a - c + d$, $a^2 + b^2$.

3. $a + b + 2c + 3d$, $ax - bx - 3c + 2d$, $ax + bx$.

4. $a^2x^2 + 2ax$, $3a + b$, $2a - c + d$, $3a^2x^2 - 4ax$.

Write the following :

5. Four positive terms ; three negative terms.

6. An expression containing three positive and two negative terms.

7. Three similar terms ; four dissimilar terms.

8. An expression containing four positive similar terms, and one containing four positive dissimilar terms.

9. An expression containing five similar terms, three of which are positive and two negative.

10. An expression containing five dissimilar terms, four of which are negative and one positive.

11. Six positive monomials ; three positive similar monomials ; four negative monomials ; three negative similar monomials.

12. Four polynomials ; four binomials ; four trinomials.

POSITIVE AND NEGATIVE QUANTITIES.

28. In arithmetic the signs $+$ and $-$ are used to indicate operations to be performed, but in algebra they are used also as **Signs of Opposition**.

Thus if *gains* are considered *positive* quantities, *losses* will be *negative*; if distances *north* from a given point are considered *positive*, distances *south* will be considered *negative*, etc.

1. Place appropriately before each of the following quantities the sign $+$ or the sign $-$:

Mr. A gains \$40 and loses \$20. Mr. S earns \$25 and spends \$15. A ship sails 40 miles north from a given parallel, and then 20 miles south. A thermometer indicates on Monday 15° above zero, and on Wednesday 10° below zero.

2. A man deposits in a bank \$50, and then draws out \$20. Indicate the transaction by using the signs $+$ and $-$.

3. Mr. H buys 25 horses and sells 10 of them; he then buys 15 more and sells 8 of them. Indicate the transactions by proper signs.

4. A vessel sailed 40 miles east, and was then driven by adverse winds 30 miles west. Indicate the directions by proper signs.

5. How much worse off is a man who is \$50 in debt than if he had nothing? Indicate his condition by the sign $+$ or $-$.

A negative quantity is sometimes regarded as less than zero.

6. Two vessels left the same port, one sailing west 10 miles per hour and the other sailing east 8 miles per hour. How far were they apart at the end of two hours? Indicate the distance sailed by each vessel and the direction.



ADDITION.

29. 1. How many days are 5 days, 4 days, and 3 days ?
2. How many d 's are 5 d , 4 d , 3 d , and 6 d ?
3. How many c 's are 6 c , 3 c , 5 c , and 2 c ?
4. How many ab 's are 3 ab , 2 ab , 4 ab , and 5 ab ?
5. When no sign is prefixed to a quantity, what sign is it assumed to have ?
6. When positive quantities are added, what is the sign of the sum ?
7. If Henry owes one boy 3 cents, another 5 cents, and another 6 cents, how much does he owe ?
8. If the sign $-$ is placed before each sum that he owes, what sign should be placed before the entire sum ?
9. What sign will the sum of negative quantities have ?
10. If a vessel sails $+5$ mi., $+8$ mi., $+9$ mi., and then sails -4 mi., -2 mi., -6 mi., how far is she from the sailing port ?
11. A boy's financial condition is represented as follows : Henry owes him 15 cents ; David owes him 10 cents ; James owes him 8 cents. He owes William 9 cents, and Fred 10 cents. What is his financial condition ? How much is 15 cents, 10 cents, 8 cents, -9 cents, -10 cents ?

30. PRINCIPLE. *Only similar quantities can be united by addition into one term.*

Dissimilar quantities cannot be added, but in algebra an *indicated* operation is often regarded as an operation performed. It is important to remember that fact.

Thus, a and b cannot be added, and yet $a + b$ is called their sum, though the operation is only indicated.

31. To add similar monomials.

1.	2.	3.	4.	5.	6.
$3x$	$9ab$	$-3mn$	$-x^2y$	ax	$-bc$
$7x$	$4ab$	$-2mn$	$-5x^2y$	$6ax$	$-7bc$
x	$3ab$	$-9mn$	$-8x^2y$	$5ax$	$-2bc$
$5x$	$6ab$	$-mn$	$-4x^2y$	$4ax$	$-12bc$
<u>$2x$</u>	<u>$2ab$</u>	<u>$-5mn$</u>	<u>$-7x^2y$</u>	<u>$3ax$</u>	<u>$-6bc$</u>

7. Express in the simplest form $5a + 3a - 2a - 7a + 12a + 3a$.

SOLUTION. The sum of the *positive* quantities is $23a$, and the sum of the *negative* quantities $-9a$. $23a - 9a = 14a$. Hence the sum of the quantities, or the simplest form of the expression, is $14a$.

Express in the simplest form :

8. $8a - 2a + a - 3a - a + 7a$.
9. $4x^2y^2 + 3x^2y^2 - x^2y^2 + 5x^2y^2 - 10x^2y^2$.
10. $7mx + 4mx - 5mx - 2mx - 6mx + mx$.
11. $5y - 3y + 8y - 10y + 6y - y$.
12. $8m + 3m - 5m - 2m + 6m - 4m$.
13. $7bc + 3bc - 4bc - 5bc + 8bc - bc$.
14. $9xy + 2xy - 5xy + 10xy - 7xy - 4xy + 5xy$.
15. $6x^2z - 4x^2z + 3x^2z + 8x^2z - 5x^2z + 3x^2z - 10x^2z$.

16. $15mn + 6mn - 10mn - 4mn - 3mn + 4mn$.
 17. $4a^2b - 3a^2b + 7a^2b - 14a^2b - 3a^2b + 20a^2b$.
 18. $25ax - 17ax - 13ax + 19ax + 6ax - 20ax$.
 19. $3(ab)^2 + 9(ab)^2 - (ab)^2 + 7(ab)^2 - 9(ab)^2$.
 20. $8(a-b) + 4(a-b) - 6(a-b) - 2(a-b) + 3(a-b)$.
 21. $7y^2z - 4y^2z + y^2z - 6y^2z + 2y^2z$.
 22. $5(x+y) - 2(x+y) - 3(x+y) + 8(x+y) - 2(x+y)$.
 23. $4(a+b)^2 + 10(a+b)^2 - 7(a+b)^2 - 2(a+b)^2 + 5(a+b)^2$.
 24. $3cd - 2cd + 5cd + 7cd - 3cd - 4cd$.
 25. $9(xy)^3 - 3(xy)^3 + 4(xy)^3 - 5(xy)^3 + 2(xy)^3 - 6(xy)^3$.

32. To add when some terms are dissimilar.

1. Add $x + 3y - z$, $x - 2y$, $x + 4y + 3z$.

PROCESS.

$$\begin{array}{r} x + 3y - z \\ x - 2y \\ x + 4y + 3z \\ \hline 3x + 5y + 2z \end{array}$$

EXPLANATION. For convenience in adding, similar terms are written in the same column, and the simplest form of the sum is obtained by beginning at either the right or left hand column and adding each column separately. The dissimilar terms in the result are connected with their proper signs.

RULE. Write similar terms in the same column. Add each column separately by finding the difference of the sums of the positive and the negative terms. Connect the results with their proper signs.

Find the sum of each of the following:

2.	3.	4.
$2a - 4b$	$10x + 3y + z$	$3xy + 2y^2 - z^2$
$6x - 2b$	$-5x - y$	$-2xy + 6y^2 - 5z^2$
$2a + 3b$	$2x - 2y + z$	$7xy - 4y^2 + 4z^2$
$-5a - 4b$	$x + 7y - 4z$	$2y^2 + 5z^2$

5. Find the sum of $2c + 5d$, $7c - d$, $d - 4c$, $2d - c$.

6. Find the sum of $6m - 4n$, $2m + 3n$, $5n - 7m$, $2n - 3m$.

Express in the simplest form :

7. $2a + 2b + 3c + 4b - 4a + 6a - 2c$.

8. $x^2z + 5xz^2 - 7xy + 6xz^2 - 2x^2z + 4xy + 4x^2z - xz^2$.

9. $3w + 4x - 7y + 2v - 2w - x + 3y + 4v - 3x + 4w - 6v$.

10. $a^2b^2 + c^2 + cd - 2c^2 - 3cd + 5a^2b^2 + cd - 3c^2 - 2a^2b^2 - c^2$.

11. Add $ab + a^2c - 5$, $3ab - 3a^2c + 7$, $2a^2c - 2ab - 3$.

12. Add $5a + 3b - 2c + d$, $2b + c - 3d$, $7a - 5b + c$.

13. Add $6m + 8n + x - y$, $2m - 2n + 3x + 4y$, $-m - 5x + 2y$.

14. Add $3x + 7y - 4z + 6w$, $7z - 4x - 2y - 5w$, $x + y + z + w$.

15. Add $4x^2y - 3xy^2 - 2x^2y^2$, $4xy^2 - 3x^2y - 2x^2y^2$, $4x^2y^3 - 2x^2y - 3xy^2$.

16. Add $3x^m + 2y^n$, $-4x^m + 5y^n$, $5x^m - 4y^n$, $7x^m - 2y^n$.

17. Add $8a^2b^2x^c - 3ab + ed$, $2a^2b^2x^c + ab - 4ed$, $2ab - a^2b^2x^c$.

18. Add $7m^2 - 6mn + 5n^2$, $4mn - 3m^2 - n^2$, $5m^2 - 4n^2 + 5mn$.

19. Add $14ax^3 - 8ay^3 + 6az^3$, $20ay^3 - 24ax^3 - 12az^3$, $32ax^3 - 40ay^3 + 15az^3$.

20. Add $10a^2b - 12a^3bc - 15b^2c^4 + 10$, $-4a^2b + 8a^3bc - 10b^2c^4 - 4$, $2a^2b + 12a^3bc + 5b^2c^4 + 2$.

21. Add $4x^3 - 6ax^2 + 5a^2x - 5a^3$, $3x^3 + 4ax^2 + 2a^2x + 6a^3$, $-17x^3 + 19ax^2 - 15a^2x + 8a^3$.

22. Add $6(c+d) - 3(c+d) + a(c+d) - 2b(c+d) - (c+d)$.

23. Add $7a - 3b + c + m$, $3b - 7a - c + m$.

24. Add $8ax + 2(x + a) + 3b$, $9ax + 6(x + a) - 9b$,
 $11x + 6b - 7ax - 8(x + a)$.

25. Add $6(x + y) + 3z - 8$, $2(x + y) - 2z + 4$, $8z - 3(x + y)$.

33. A letter may sometimes represent some definite number.

Thus, a may represent 5 ; then $2a$ will represent 10 ; $3a$, 15, etc.

A letter may also represent any number, whatever its value.

Thus, 5 times n may represent 5 times any number ; 8 times n or $8n$ may stand for 8 times any number.

34. Letters used to represent quantities having known values, or letters which represent any number or quantity are called **Known Numbers or Quantities**.

The *first letters* of the alphabet, as a , b , c , etc., are used to represent *known numbers or quantities*.

Thus, a , b , c , d , etc., are usually considered known quantities ; that is, they stand either for known numbers or for any number.

PROBLEMS.

35. 1. A man bought a barrels of flour, b barrels of sugar, and 8 barrels of molasses. How many barrels in all did he buy ?

2. Edith bought a ribbon for m cents, a pencil for d cents, and a book for 6 cents. How many cents did she pay for all ?

3. A farmer sold some sheep for c dollars, a cow for n dollars, and a horse for as much as he received for the sheep and cow. How much did he receive for all ?

4. George walked a miles, he then rode 3 miles on his bicycle, and b miles on the cars. How far did he travel ?

5. A man began business with $2c$ dollars. The first year he gained $\frac{1}{2}$ as much as he had; the second year $\frac{1}{3}$ as much as he had at the end of the first year; and the third year \$400. How much did he gain in the three years?

6. The letter b represents an odd number. What will represent the next even number? What the next odd number?

7. Laura is m years old; Lizzie is twice as old as Laura; and Mabel's age is equal to $\frac{1}{2}$ the sum of the ages of the other two. What is the sum of their ages?

8. A merchant took in c dollars one week, d dollars the next, and \$75 the next. How many dollars did he receive in the three weeks?

9. What is the sum of seven terms of $x + x + x + \text{etc.}$? Of a terms of $x + x + x + \text{etc.}$?

10. A grocer sold b pounds of sugar, c pounds of coffee, d pounds of tea, and 2 pounds of chocolate. How many pounds of groceries did he sell?

11. A man paid m dollars for a coat, n dollars for a waistcoat, p dollars for trousers, g dollars for a pair of boots, and r dollars for a hat. How much did his outfit cost him?

12. A man paid a dollars for a farm; he then expended upon improvements d dollars, and sold it for b dollars more than the entire cost. How much did he receive for it?

13. My fare to San Francisco was a dollars, my sleeping-car charges c dollars, my meals cost me b dollars, and my other expenses \$25. How much did I expend before I reached San Francisco?

SUBTRACTION.

36. 1. What is the difference between 8 days and 5 days?

2. What is the difference between 11 cents and 4 cents?

3. What is the difference between $12d$ and $5d$?

4. What is the difference between $15c$ and $8c$?

5. What is the difference between $12f$ and $8f$?

6. Subtract:

$8x$ from $13x$; $4y$ from $15y$; $6c$ from $13c$; $8d$ from $20d$;
 $10x^2y$ from $14x^2y$; $8x^3y^2$ from $21x^3y^2$; $4xy^2$ from $14xy^2$;
 $10a^2b^2$ from $18a^2b^2$; $8a^2x^2y^2$ from $15a^2x^2y^2$; $4abcd$ from
 $13abcd$; $6abcx$ from $18abcx$; $10ax^2y^2z^2$ from $20ax^2y^2z^2$.

7. What is the remainder when $4a^2x^2y^2$ is subtracted from $10a^2x^2y^2$? What is the sum of $10a^2x^2y^2$ and $-4a^2x^2y^2$?

8. What is the remainder when $3abc^2$ is subtracted from $12abc^2$? What is the sum of $12abc^2$ and $-3abc^2$?

9. Instead of subtracting a positive quantity, what may be done to secure the same result?

10. What is the result when 8 is subtracted from 15? What, when $8 - 5$ is subtracted from 15?

11. Why is the result 5 more in the latter case than in the former?

12. What is the result when $10a$ is subtracted from $18a$? What, when $10a - 8a$ is subtracted from $18a$?

13. Why is the result greater by $8a$ in the latter case than in the former?

14. Instead of subtracting a negative quantity, what may be done to secure the same result?

15. One vessel was 40 miles east and another 20 miles west from a given meridian. Indicate their relations by proper signs. How far apart were they?

16. Mr. A's property is worth $\$25a$ and Mr. B is $\$8a$ in debt. Indicate their financial conditions by proper signs. What was the difference in their financial condition?

17. A thermometer indicated a temperature of 35° above zero on Jan. 5, and of 15° below zero on Jan 6. Indicate the temperature by proper signs. What was the difference in temperature?

37. PRINCIPLES. 1. *The difference between similar quantities, only, can be expressed in one term.*

2. *Subtracting a positive quantity is the same as adding a numerically equal negative quantity.*

3. *Subtracting a negative quantity is the same as adding a numerically equal positive quantity.*

38. To subtract when the terms are positive.

1. From $10a$ subtract $4a$.

PROCESS.	EXPLANATION.
$\begin{array}{r} 10a \\ 4a \\ \hline 6a \end{array}$	When four times any number is taken from ten times that number, the remainder is six times the number; therefore, when $4a$ is subtracted from $10a$, the remainder is $6a$. Or, since subtracting a positive number or quantity is the same as adding an equal negative quantity (Prin. 2), $4a$ may be subtracted from $10a$ by changing the sign of $4a$ and adding the quantities. Therefore, to subtract $4a$ from $10a$, we find the sum of $10a$ and $-4a$, which is $6a$.

2. From $13m$ take $15m$.

PROCESS. **EXPLANATION.** After subtracting from $13m$ as much as we can of $15m$, there will be $2m$ yet to be subtracted, or the result will be $-2m$. Or, since subtracting a positive number or quantity is the same as adding an equal negative quantity (Prin. 2), $15m$ may be subtracted from $13m$ by finding the sum of $13m$ and $-15m$, which is $-2m$. Therefore, when $15m$ is taken from $13m$, the result is $-2m$.

	3.	4.	5.	6.	7.	8.
From	$19x$	$7ab$	$18m^2$	$16xy$	$6x^2y^2z$	$9mn$
Take	<u>$4x$</u>	<u>$3ab$</u>	<u>$13m^2$</u>	<u>$20xy$</u>	<u>$8x^2y^2z$</u>	<u>$14mn$</u>

Subtract the following:

9. $8x + 3y$ from $12x + 7y$.
10. $5ab + 3c$ from $10ab + 2c$.
11. $4b + 9d$ from $8b + 5d$.
12. $5x + y$ from $8x + 3y$.
13. $4x + 8z$ from $5x + 9z$.
14. $3a + 2b + 5c$ from $7a + 5b + 6c$.
15. $7a + 2b$ from $9a + b$.
16. $5x^2 + 8y^2 + 6xy$ from $7x^2 + 3y^2 + 2xy$.
17. $4xy + 3z$ from $10xy + 5z$.
18. $5(a^2 + b^2) + 7c^2 + 2d$ from $6(a^2 + b^2) + 4c^2 + 4d$.
19. $3a(p + q) + 2$ from $6a(p + q) + 7$.
20. $5x + 7y + 4z + 5$ from $5x + 4y + 8z + 6$.
21. $3x^2 + 2y + z + 4v$ from $7x^2 + 3y + 6z + v$.
22. $2x + 5(y + z) + 8$ from $7x + 2(y + z) + 10$.
23. $3a^2 + 5b^2 + 4c^2 + d^2$ from $a^2 + 6b^2 + c^2 + 5d^2$.

39. To subtract when some terms are negative.1. From $7x - 2y$ subtract $4x - 3y$.

PROCESS. EXPLANATION. Since the subtrahend is composed of two terms, each term must be subtracted separately.

$7x - 2y$ Subtracting $4x$ from $7x - 2y$, leaves $3x - 2y$, or the

$4x - 3y$ result may be obtained by adding $-4x$ to $7x - 2y$.

$- \quad +$ But since the subtrahend was $3y$ less than $4x$, to obtain

$3x + y$ the true remainder, $3y$ must be added to $3x - 2y$, which gives $3x + y$.

Therefore the subtraction may be performed by changing the sign of each term of the subtrahend and adding the quantities.

RULE. Write similar terms in the same column. Change the sign of each term of the subtrahend from $+$ to $-$, or from $-$ to $+$, or conceive it to be changed, and proceed as in addition.

	2.	3.	4.	5.	6.
From	$4a^3x$	$9x^2yz$	$6y - 4z$	$7x^2 + 5y^2$	$3a - 3b + c$
Take	<u>$3a^3x$</u>	<u>$-7x^2yz$</u>	<u>$5y + 2z$</u>	<u>$4x^2 - 8y^2$</u>	<u>$3a + 3b - c$</u>

	7.	8.	9.
From	$4ab - 3c + d$	$5x^2 - 3xy + x$	$10ax - 13y + z^2$
Take	<u>$-2ab + 5c - d$</u>	<u>$6x^2 + 5xy - x$</u>	<u>$5ax - 7y - z^2$</u>

10. From $12x^5 - 20x^3y^3 + 6xy$ subtract $9x^3y^3 - 2xy + 4x^5$.11. From $4ab + 3b^2 - 6cd$ subtract $6ab - 2b^2 + 2cd$.12. From $9x^2y - 6xy^2 - 2xy + 5$ subtract $3x^2y + xy + 8$.13. From $7x^m + 2x^my^n - 5y^n$ subtract $4x^m - 2x^my^n - 9y^n$.14. From $8(m + n^2) - 12(m^2 + n)$ subtract $12(m + n^2) - 8(m^2 + n)$.15. From $4x^3 + 7y^3 - 3z^3 + r^3$ subtract $-2x^3 + y^3 - 3r^3$.16. From $5(p + q) - 6(r + s) + 15$ subtract $8(p + q) + 2(r + s) + 25$.

17. From $18m^2nx^2 + 12a^2bc^2 + abd$ subtract $-2m^2nx^2 + 5abd$.

18. From $a^2 - 2ab + c^2 - 3b^2$ subtract $2a^2 - 2ab + 3b^2$.

19. From $2x^2 + 2y^2 - 4xy$ subtract $2x^2 + 2y^2 + 4xy$.

20. From $4x^4 + 3x^2y^2 - 8y^4$ subtract the sum of $x^4 - x^2y^2 + 3y^4$ and $2x^4 + 2x^2y^2 - 9y^4$.

21. From $5x^3y^3 + 10x^4y - 6yz^3$ subtract $10x^4y - 4x^3y^3 + 5yz^3$.

22. From $3x^m - 4x^ny^m + 4y^m$ subtract $4x^m + 2x^ny^m - y^m$.

23. From $4bx^3 + 3ay^2 + 4 - cy$ subtract $cy - 5 - bx^3$.

24. From $10m^2 - 4mn - 3n^2 - 18$ subtract the sum of $m^2 - 3mn + 4$ and $5m^2 - 2mn + 6n^2$.

25. From $a^2x - x^2 + x^3y - 8$ subtract $3a^2x - 5 - x^2 + 2x^3y$.

26. From $15x^2y^2 - 15$ subtract $4x^2y^2 + z^2 - 4y^2 - 10$.

27. From $2x + y + z + u$ subtract $x + 2y + 2z + 2u$.

28. From $16a^2b^2 - 12bc + 14b^2$ subtract the sum of $a^2b^2 - 3bc + 4$ and $10a^2b^2 + 8bc + 5b^2$.

29. From $ax + by$ subtract $cx - dy$.

PROCESS.

$$ax + by$$

$$cx - dy$$

$$(a - c)x + (b + d)y$$

EXPLANATION. Since a and c may be regarded as the coefficients of x , and b and $-d$ the coefficients of y , the difference between the quantities may be found by writing the difference between the coefficients of x and y respectively for the coefficients of the remainder. Since c cannot be subtracted from a , the

subtraction is indicated by $(a - c)$, and since $-d$ cannot be subtracted from b , the subtraction is indicated by $(b + d)$, consequently the remainder may be written $(a - c)x + (b + d)y$.

30. From $ax + 2y$ subtract $2x - by$.

31. From $2cx - ay + 3z$ subtract $cy + 5x - az$.

32. From $cx - 12aby + 4a^2b^2$ subtract $6x - 10aby + 3b^2$.
33. From $2a(x - y) + 4abx$ subtract $3c(x - y) + 2ax$.
34. From $ax + by + cz$ subtract $bx + cy + dz$.
35. From $3x + 7y + 8z$ subtract $bcx - ay + az$.
36. From $5ay + 2cz + 6x$ subtract $cy - az - dx$.
37. From $(a - b)x + (c + d)y$ subtract $ax + dy$.
38. From $7x^2 + 5ay^2 + 6z^2$ subtract $2ax^2 + 3y^2 - abz^2$.
39. From $(m + n)x^2 + (m - n)y^2 + z^2$ subtract $mnx^2 - 4y^2 + az^2$.
40. From $a(x + y) + b(x - y) + dx$ subtract $b(x + y) - a(x - y) - cx$.
41. Express the difference between m and n .
42. Write the number one less than x ; the number one greater.
43. A man sold a horse for \$125 and gained a dollars. What did the horse cost?
44. A girl earns b cents a day and spends 25 cents a week. How much has she left at the end of the week?
45. The sum of two numbers is 30, and x represents one of them. What represents the other?
46. The difference between two numbers is 5. How may the numbers be represented?
47. A merchant bought a hat for b dollars and a coat for c dollars, and sold the two for d dollars. What represents his gain?
48. A man whose income is a dollars spends m dollars for rent, n dollars for living expenses, and 100 dollars for other expenses. What represents the amount he saves?

49. A lady paid a dollars for a dress, c dollars for a hat, and \$25 for a cloak. How much had she left from a \$50 bill?

50. A man paid 40 dollars for a cords of wood, and sold it at 3 dollars a cord. How much did he gain?

51. A farmer sold some grain for b dollars, some fruit for d dollars, and some hay for e dollars. He received in part payment a horse worth f dollars. How much remained still to be paid?

40. The parenthesis, $()$, the vinculum, $—$, the brackets, $[]$, and the braces, $\{ \}$, are called **Signs of Aggregation**. They show that the quantities included by them are to be treated as a single quantity.

Thus, $(a + b) \times c$, $\overline{a + b} \times c$, and $\{a + b\} \times c$ show that the sum of a and b is to be multiplied by c .

41. The subtrahend is sometimes expressed with a sign of aggregation and written after the minuend with the sign $-$ between them.

Thus, when $b + c - d$ is subtracted from $a + b$, the result is sometimes indicated as follows: $a + b - (b + c - d)$.

1. What change must be made in the signs of the terms of the subtrahend when it is subtracted from the minuend?

2. When a quantity in parenthesis is preceded by the sign $-$, what change must be made in the signs of its terms when the subtraction is performed, or when the parenthesis or other similar sign is removed?

The term parenthesis is commonly used to include all signs of aggregation.

42. PRINCIPLES. 1. *A parenthesis, preceded by the minus sign, may be removed from an expression if the signs of all the terms in parenthesis are changed.*

2. *A parenthesis, preceded by the minus sign, may be used to inclose an expression if the signs of all the terms to be inclosed in parenthesis are changed.*

When quantities are inclosed in a parenthesis preceded by the plus sign, the parenthesis may be removed without any change of signs, and, consequently, any number of terms may be inclosed in a parenthesis with the *plus* sign without any change of signs.

The student should remember that in expressions like $-(x^2 - y + z)$ the sign of x^2 is plus, and the expression is the same as if written $-(+x^2 - y + z)$.

Simplify the following:

1. $20 - (3 + 4 - 6 + 5)$.

SOLUTION. $20 - 3 - 4 + 6 - 5 = 14$.

2. $25 - (6 + 9 - 3 - 7 + 13)$.

3. $5 - (3 + 2 - 6) - (-2 - 4 + 1)$.

4. $(7 - 5 + 3) - (8 - 9 + 2)$.

5. $15 - (3 + 4 - 6) - (8 - 6 + 7)$.

6. $(4 + 10 - 8) - (3 + 7 - 4)$.

7. $19 - (12 - 5 + 8) - (11 - 5 - 3)$.

8. $30 - (-6 + 8 - 1) - (18 - 10 + 4)$.

9. $(14 - 5 + 2) - (18 - 20 + 5)$.

10. $15 - (3 + 8 - 7) + 16 - (10 + 4 + 5 - 8)$.

11. $(5 + 7 - 8) - 6 - (-5 + 3 - 6 + 2)$.

12. $16 - [8 - (5 + 6 - 4) + 12]$.

SOLUTION. $16 - [8 - (5 + 6 - 4) + 12]$
 $= 16 - [8 - 5 - 6 + 4 + 12]$
 $= 16 - 8 + 5 + 6 - 4 - 12$
 $= 3$.

$$13. 25 - [13 - 4 + (3 - 10 - 2)].$$

$$14. 10 - \{8 - (15 + 7 + 3) + 6\}.$$

$$15. 17 - (3 + 8) - [12 - (3 + 8) - 5].$$

$$16. -\{-16 + 13 - (6 - 1 + 4) + 5 - 10\}.$$

$$17. (3 + 7 - 4) - [14 - (13 + 7) + 5].$$

Simplify the following:

$$18. a - (b - c + d - e).$$

$$\text{SOLUTION. } a - (b - c + d - e) = a - b + c - d + e.$$

$$19. 2x - (x - 5x + 3x - 8x).$$

$$20. 7m - (3n + 2m - 6m + n).$$

$$21. 4x^2 + 7ax - (5ax + 3ax - 2x^2 + 10ax).$$

$$22. a^2 + b^2 - (-2ab - 2a^2 - 2b^2).$$

$$23. (6xy + 2z) - (4z + 3xy - 2z + 5).$$

$$24. a - b - (a + b - c - 3).$$

$$25. a + b - (2a - 3b) - (5a + 7b) - (-13a + 2b).$$

$$26. (a + b + c) + (-a + b - c) - (a - b + c).$$

$$27. x - [-x + 2x - (x + 2x) - 2x].$$

$$28. 3x - [x - 3z - (2y - z)].$$

$$29. a^2 - a - (4a - y - 3a^2 - 1).$$

$$30. m + n - (m + n) - \{m - n - (m + n) - n\}.$$

$$31. (x^2 + 2xy + y^2) - (2xy - x^2 - y^2).$$

$$32. 9x - [8x - (6x - 3x)].$$

$$33. (x + 10) - \{x - (3x + 25) - 10\}.$$

$$34. 8a - (6a - 5) - (5a + 11 - 4a).$$

$$35. a - [2b - (3c + 2b) - a].$$

TRANSPOSITION IN EQUATIONS.

43. 1. If a certain number, diminished by 3, equals 15, what is the number? If $x - 3 = 15$, what is the value of x ?

2. If a certain number, increased by 3, equals 15, what is the number? If $x + 3 = 15$, what is the value of x ?

3. In the equation $x - 3 = 15$, what is done with the 3 in obtaining the value of x ? In the equation $x = 15 + 3$, how does the sign of 3 compare with its sign in the original equation?

4. In the equation $x + 3 = 15$, what is done with the 3 in obtaining the value of x ? In $x = 15 - 3$, how does the sign of 3 compare with its sign in the original equation?

5. In changing the 3's from one side, or *member*, of the equation to the other, what change was made in the sign?

6. When a number or quantity is changed from one member of an equation to the other, what change must be made in its sign?

7. If any number, as 5, is added to one member of the equation $2 + 3 = 5$, what must be done to the other member to preserve the equality?

8. If any number, as 3, is subtracted from one member of the equation $2 + 3 = 5$, what must be done to the other member to preserve the equality?

9. If one member of the equation $2 + 3 = 5$ is multiplied by any number, as 4, what must be done to the other member to preserve the equality?

10. If one member of the equation $2 + 3 = 5$ is divided by any number, as 5, what must be done to the other member to preserve the equality?

11. If one member of the equation $5 + 3 = 8$ is raised to any power, as the second power, what must be done to the other member to preserve the equality?

12. What, then, may be done to the members of an equation without destroying the equality?

44. The parts on each side of the sign of equality are called the **Members of an Equation**.

45. The part of an equation on the left of the sign of equality is called the **First Member**.

46. The part of an equation on the right of the sign of equality is called the **Second Member**.

47. The process of changing a term from one member of an equation to another is called **Transposition**.

48. A truth that does not need demonstration is called an **Axiom**.

AXIOMS. 1. *Things that are equal to the same thing are equal to each other.*

2. *If equals are added to equals, the sums are equal.*

3. *If equals are subtracted from equals, the remainders are equal.*

4. *If equals are multiplied by equals, the products are equal.*

5. *If equals are divided by equals, the quotients are equal.*

6. *Equal powers of equal quantities are equal.*

49. PRINCIPLE. *A term may be transposed from one member of an equation to the other if its sign is changed from + to -, or from - to +.*

EQUATIONS AND PROBLEMS.

50. 1. Given $2x - 3 = x + 6$, to find the value of x .

PROCESS.

$$\begin{array}{r} 2x - 3 = x + 6 \\ + 3 = + 3 \\ \hline \end{array}$$

$$2x = x + 9$$

$$x = x$$

$$x = 9$$

OR,

$$2x - 3 = x + 6$$

$$2x - x = 6 + 3$$

$$x = 9$$

VERIFICATION.

$$18 - 3 = 9 + 6$$

$$15 = 15$$

EXPLANATION. Since the known and the unknown quantities are found in both members of the equation, to find the value of x , the known quantities must be collected in one member and the unknown in the other.

Since -3 is found in the first member, it may be caused to disappear by adding 3 to both members (Ax. 2), which gives the equation, $2x = x + 9$.

Since x is found in the second member, it may be caused to disappear by subtracting x from both members (Ax. 3), which gives as a resulting equation, $x = 9$.

Or, since a term may be changed from one member of an equation to the other if its sign is changed (Prin.), -3 may be transposed to the second member if it is changed to $+3$, and x may be transposed to the first member if it is changed to $-x$. Then, the resulting equation will be $2x - x = 6 + 3$.

By uniting the terms, $x = 9$.

The result may be *verified* by substituting the *value* of x for x in the original equation.

If both members are then *identical*, the value of the unknown quantity is correct. Thus, if 9 is substituted for x in the original equation, the equation becomes $18 - 3 = 9 + 6$, or $15 = 15$.

Therefore, 9 is the correct value of x .

RULE. *Transpose the terms so that the unknown terms stand in the first member of the equation, and the known terms in the second.*

Unite similar terms, and divide each member of the equation by the coefficient of the unknown quantity.

VERIFICATION. *Substitute the value of the unknown quantity for the quantity in the original equation. If the members are then identical in value, the value of the unknown quantity found is correct.*

Transpose, and find the value of x :

2. $x + 4 = 10$.
3. $x - 3 = 4$.
4. $2x + 1 = 5$.
5. $6x - 6 = 12$.
6. $4x + 3 = 15$.
7. $8x - 2 = 14$.
8. $3x + 5 = 26$.
9. $9x - 5 = 31$.
10. $5x + 2 = 10 + 7$.
11. $7x - 1 = 30 + 4$.
12. $2x - 10 = 3 + 5$.
13. $4x - 2x = 3 + 7$.
14. $6x - 3 = 2x + 9$.
15. $5x - 15 + 2 = 2 - 3x + 1$.
- 16. $6x - 2x + 4 = 16 - 8$.
17. $7x - 3 = 2x - 4 + 11$.
18. $9x - 10 = 2x + 4$.
19. $6x + 25 = 88 - x$.
20. $3x - 4 = 12 - 4$.
21. $5x - 5 = 67 - 3x$.
22. $10x - 20 = 24 - 12x$.
23. $3x - 14 = 10 - x$.
24. $2x - 16 = 20 - 4x$.
25. $15x - 39 = 29 - 2x$.
26. $3x - 18 = 31 - 4x$.
27. $4x - 14 = 49 - 3x$.
28. $5x - 20 = 25 - 4x$.
29. $2x - 36 = 60 - 6x$.
- 30. $3x - 20 + x = 44 - 4x$.
31. $5 + 8x - 7 = 3x + 3$.
32. $9x - 15 = 6 + 7x + 3$.
33. $9x + 5 - 15 = 5x + 2x$.
34. $8x - 10 = 10 + 2x + 4$.
35. $4x - 6x - 6 = 6 + 3 - 3x$.
36. $x - 5 = 18 - 4x - 3$.
37. $3x + 6x = 18 - x + 2$.
38. $3x - 6 = x + 14 - 4$.
39. $9x + 13 = 26 + 2x + 1$.
40. $4x + 4 - 3x = 16 - 2x$.
41. $27x - 14 = 190 - 41x$.
42. $6x - 12 = 4x + 18$.
43. $5x - 15 = 3x + 9$.
44. $18x + 9 = 15x + 30$.
45. $7x - 3 + 2x = 5x + 20 + 1$.
46. $6x - 21 = x + 14 - 2x$.
- 47. $9x + 3 - 24 = 5x - 5$.
48. $10x - 4 + 3 = 6x + 19$.
49. $3x - 15 - 10 = 20 - 2x$.
50. $5x - 5 - 20 + 6x = 41$.
51. $2x - 25 = 35 - x - 3x$.
52. $3x - 19 = 20 - 10x + 13$.
53. $5x - 16 = 25 - x + 40 - 3x$.
54. $6x - 5 - 30 = 10 - 4x - 5x$.
55. $7x - 30 = 10 + 16 - 7x$.
56. $5x - 50 = 25 - 5x + 25$.
57. $10x - 22 = 17 - 2x - x$.

PROBLEMS.

51. 1. Twice a certain number increased by 15 is equal to the number increased by 19. What is the number?

2. What number is that whose double exceeds the number by 12?

3. Ten times a certain number diminished by 13 is equal to the number plus 5. What is the number?

4. What number diminished by 8 equals 6?

5. Six times a certain number, plus 7, equals five times the number, plus 12. What is the number?

6. Three boys together had 90 cents. The first had 10 cents more than the second, and the second had 1 cent more than the third. How much had each?

7. A father gave a certain sum to his youngest son, and 4 cents more to the next older, and 10 cents to the oldest. If he gave to all 20 cents, how much did he give to each?

8. The greater of two numbers exceeds the less by 14, and the sum of the numbers is 34. What are the numbers?

9. A and B started in business, A furnishing \$4000 more than B. Three times B's capital was then equal to A's. How much did each furnish?

10. A and B had the same sum of money. A gave B \$4, and then B had double the amount A had left. How much had each at first?

11. A tourist rode 32 miles upon a bicycle. A certain number of miles was downhill, twice as far plus 8 miles was level, and the distance uphill was $\frac{1}{2}$ as far as the distance on a level. How many miles did he travel upon each kind of road?

12. A man has two horses, of unequal value, together worth \$200. If he should put a saddle worth \$30 on the poorer horse, the horse and saddle would together be equal in value to the better horse? What is the value of each?

13. Six hundred gallons of water are discharged into a cistern by 3 pipes. The second discharges 100 gallons more than the first, and the third discharges three times as much as the first. How many gallons are discharged by each pipe?

14. A drover being asked the number of his cattle said that if he had three times as many as he then had and 25 more, he would have 1000. How many cattle had he?

15. A man bought a watch and chain for \$60. The watch cost 12 times as much as the chain lacking \$5. What was the cost of each?

16. A tenement house contained 90 persons, men, women, and children. If there were 4 more men than women, and 10 more children than men and women together, how many were there of each?

17. A steamer and its cargo are together worth \$120,000. If the steamer lacks only \$8400 of being worth twice as much as the cargo, what is the value of each?

18. A clerk's expenses are \$400 per year, and his brother's are \$600 per year. If the brother has three times as large an annual salary and he has left at the end of the year a sum equal to twice the clerk's salary, what is the salary of each?

19. If a house and lot cost four times as much as the lot, and the house cost \$2500 more than twice as much as the lot, what was the cost of each?

MULTIPLICATION.



52. 1. What is the sum of $5m + 5m + 5m$? Or, how much is 3 times $5m$? 8 times $5m$?

2. What is the sum of $8xy + 8xy + 8xy + 8xy$? Or, how much is 4 times $8xy$? 10 times $8xy$?

3. How much is 6 times $7bc$? Which quantity is the multiplier? Which is the multiplicand? What sign has the multiplier? What sign has the multiplicand? What sign has the product?

4. When a *positive* quantity is multiplied by a *positive* quantity, what is the sign of the product?

5. If a vessel sails south 8 miles per hour, indicated by -8 mi., how far will she sail in 5 hours? What will be the sign of the product?

6. How much is 4 times $-5xy$? 6 times $-6ab$? 7 times $-8cd$? What is the sign of the multiplier in each case? What is the sign of the multiplicand? What is the sign of the product? $-4ab \times 8 = ?$

7. When a *negative* quantity is multiplied by a *positive* quantity, what is the sign of the product?

8. How does the product of 6 times 7 compare with the product of 7 times 6? What effect upon the product has it to change the order of the factors, when the numbers or quantities are abstract?

9. How, then, will the product of $-3a$ multiplied by $+4$ compare with the product of $+4$ multiplied by $-3a$? What is the product? $-5xy \times 6 = ?$ $6 \times -5xy = ?$ $-7ab \times 4 = ?$ $4 \times -7ab = ?$

10. When a *positive* quantity is multiplied by a *negative* quantity, what is the sign of the product?

11. How much is 6 times $-3a$? 2 times $-3a$? $(6-2)$ times $-3a$?

What will be the sign of $-6a$ after it is subtracted?

12. Since -2 times $-3a$ gives a product of $+6a$, what may be inferred regarding the algebraic sign of the product when a *negative* quantity is multiplied by a *negative* quantity?

13. What is an exponent? What does it show? What does a^5 mean? When a^5 is multiplied by a^2 , how many times is a used as a factor to obtain the product? How many times, when a^4 is multiplied by a^5 ? When a^7 is multiplied by a^8 ?

14. How, then, may the number of times a quantity is used as a factor in multiplication be determined from the exponents of the quantities in the expressions multiplied? How may the exponent of a quantity in the product be determined?

15. $3a^2 \times 6 = ?$ $10a^5 \times 5 = ?$ $20a^3 \times 3 = ?$ $25x^2y \times 2 = ?$ How is the coefficient of the product determined from the coefficients of the factors, or from the multiplier and the multiplicand?

53. Multiplication is indicated in four ways:

1. By the sign $\times$, read *multiplied by* or *times*.

Thus, $a \times b$ shows that a is to be multiplied by b .

2. By the dot ($\cdot$), read *multiplied by* or *times*.

Thus, $a \cdot b$ shows that a is to be multiplied by b .

3. By writing letters, or a number and a letter, side by side.

Thus, ab shows that a is to be multiplied by b ; and $5a$ shows that a is to be multiplied by 5.

4. By a small figure or letter, called an *Exponent*, written a little above and at the right of a quantity, showing the number of times the quantity is to be used as a factor.

Thus, a^5 shows that a is to be used as a factor 5 times, or that it is equal to $a \times a \times a \times a \times a$ or $aaaaa$.

54. PRINCIPLES. 1. *The sign of the product of two factors is + when they have LIKE signs, and - when they have UNLIKE signs.*

2. *The coefficient of the product is equal to the product of the coefficients of the factors.*

3. *The exponent of a quantity in the product is equal to the sum of its exponents in the factors.*

55. To multiply when the multiplier is a monomial.

1. What is the product of $5x^2yz$ multiplied by $3abx$?

PROCESS.

$$\begin{array}{r} 5x^2yz \\ 3abx \\ \hline 15abx^3yz \end{array}$$

EXPLANATION. The coefficient of the product is obtained by multiplying 5 by 3 (Prin. 2). The literal quantities are multiplied by adding their exponents (Prin. 3). Hence, the product is $15abx^3yz$.

2. What is the product of $3b - c$ multiplied by $5c^2$?

PROCESS.

$$\begin{array}{r} 3b - c \\ 5c^2 \\ \hline 15bc^2 - 5c^3 \end{array}$$

EXPLANATION. The product of $3b$ multiplied by $5c^2$ is $15bc^2$. But, since the *entire* multiplicand is $3b - c$, the product of c multiplied by $5c^2$ must be subtracted from $15bc^2$. The product of c multiplied by $5c^2$ is $5c^3$, which subtracted from $15bc^2$ gives the entire product $15bc^2 - 5c^3$.

RULE. *Multiply each term of the multiplicand by the multiplier, as follows:*

To the product of the numerical coefficients, annex each literal factor with an exponent equal to the sum of the exponents of that letter in both factors.

Write the sign + before each term of the product when its factors have like signs, and — when they have unlike signs.

	3.	4.	5.	6.	7.	8.	9.	10.
Multiply	10	$10a$	$-6a$	$22x$	18	$-14b$	$17c$	$24x$
By	4	-4	3	-5	$6x$	3	$3c$	$-9x$

	11.	12.	13.	14.	15.
Multiply	$5x^2yz$	$-15xy^2$	$16c^2dz$	$-42m^2n^2$	$-25a^2b$
By	$2x^2yz$	$-3xy^2$	$-4dz$	$-3mn$	$5ab^2$

	16.	17.	18.	19.
Multiply	$(a+b)$	$4(x-y)$	$3x+2y-5z$	$2x^2y-2y^2z$
By	4	-5	$2a$	$3x$

Multiply:

20. $3a^2x - 5x^2y + 2y^2$ by $-4xy$.

21. $a^2 + 2ab + b^2$ by ab .

22. $6m + 7mn + 5n^2$ by $-3mn$.

23. $x^4 - 2x^3 + 5x^2 + x - 3$ by $9x^2$.

24. $9a^2 - 13ab + 4b^2 - 6$ by $12a^2b^2$.

25. $x^2 - 3xy - 3xz + y^2z^2$ by $2ax$.

26. $4ab + 3a^2b - 5ab^2 - 2a^2$ by $3b^2$.

27. $3m^2 - 10mn - 8n^2$ by $4mn$.

28. $x^4 + x^3 + x^2 + x + 1$ by $-5x$.

29. $6a^2 - 18ab + 15c^2 - 20abc + 14$ by $3a^2b^2$.

56. To multiply when the multiplier is a polynomial.

1. Multiply $a + b$ by $a + b$.

SOLUTION.

$$\begin{array}{r} a + b \\ a + b \\ \hline a \text{ times } a + b = a^2 + ab \\ b \text{ times } a + b = \quad ab + b^2 \\ \hline (a + b) \text{ times } (a + b) = a^2 + 2ab + b^2 \end{array}$$

RULE. *Multiply each term of the multiplicand by each term of the multiplier, and add the partial products.*

$$\begin{array}{r} 2. \\ 2ab - 3c \\ 4ab + c \\ \hline 8a^2b^2 - 12abc \\ \quad 2abc - 3c^2 \\ \hline 8a^2b^2 - 10abc - 3c^2 \end{array}$$

$$\begin{array}{r} 3. \\ x^3 - 3x^2y + 3xy^2 - y^3 \\ x - y \\ \hline x^4 - 3x^3y + 3x^2y^2 - xy^3 \\ \quad - x^3y + 3x^2y^2 - 3xy^3 + y^4 \\ \hline x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 \end{array}$$

Multiply:

- | | |
|-----------------------------------|----------------------------------|
| 4. $x + y$ by $x + y$. | 17. $5m - 4n$ by $4m + 5y$. |
| 5. $m + n$ by $m - n$. | 18. $x + 2y$ by $x + 5y$. |
| 6. $a^2 + 2ab + b^2$ by $a + b$. | 19. $1 + x + x^2$ by $1 - x$. |
| 7. $2a - 5b$ by $2a + 5b$. | 20. $x + y + 1$ by $x - y - 1$. |
| 8. $x + 4$ by $x - 10$. | 21. $2x + 4y$ by $3x - 2y$. |
| 9. $3y + 2z$ by $2y + 3z$. | 22. $a + b - 2c$ by $2a - b$. |
| 10. $3a + 7b$ by $3a - 7b$. | 23. $4x + 7$ by $3x - 2$. |
| 11. $2x + 1$ by $3x - 6$. | 24. $3am + 6c$ by $8ac + c^2$. |
| 12. $2a - 3b$ by $3a + 5b$. | 25. $3xy - 6y$ by $4xy + 8y$. |
| 13. $3m + 4n$ by $2m + 3n$. | 26. $x + y - z$ by $x + y$. |
| 14. $5y - 3z$ by $4y - 4z$. | 27. $a - b - c$ by $a - c$. |
| 15. $2b - 5c$ by $3b + 8c$. | 28. $2a + x - y$ by $a - x$. |
| 16. $3x - 20$ by $8x + 4$. | 29. $2x + 3y - 6$ by $x + 5y$. |

30. $3xz + 2y^2$ by $8xz - 4y^2$.
31. $m + n + 1$ by $m - n + 1$.
32. $5x + 4$ by $5x - 9$.
33. $3x^2 - 4y^2$ by $3x^2 - 7y^2$.
34. $a^3 + 3a^2b + 3ab^2 + b^3$ by $a + b$.
35. $4x^2 - 12xy + 9y^2$ by $2x - 3y$.
36. $m^4 + m^3n + m^2n^2 + mn^3 + n^4$ by $m - n$.

PROBLEMS.

57. 1. If from three times a number 4 is subtracted and the remainder is multiplied by 6, the result is 12. What is the number?

2. If from two times a number 4 is subtracted and the remainder is multiplied by 3, the result equals two times the sum of that number and 2. What is the number?

3. A father is four times as old as his son, and 5 years ago he was seven times as old as his son. What is the age of each?

4. Samuel and John together have 40 cents. If John had 5 cents less, and Samuel 5 cents more, Samuel would have three times as much money as John. How many cents has each?

5. A commenced business with three times as much capital as B. During the first year A lost $\frac{1}{3}$ of his money, and B gained \$500. The amount of A's and B's money was then equal. How much had each at first?

6. A is 50 years of age; B is 10. When will A be three times as old as B?

7. Six men hired a boat, but 2 of them being unable to pay their share, the other 4 were obliged to pay 1 dollar more each. For how much did they hire the boat?

8. Three times the difference between a certain number and 10 equals two times the sum of the number and 10. What is the number?

9. Express the product of the factors $2, x, y, z, x^2, y, 4z$.

10. What will d quarts of milk cost at f cents per quart?

11. How far will a man travel in a hours if he goes $b+6$ miles per hour?

12. A farmer has a cows and three times as many sheep less 8. How many animals does he own?

13. A man sold 20 acres of land at a dollars per acre. With a part of the money he bought 3 horses at d dollars each. How much money had he left?

14. If a men can do some work in 12 days, how long will it take one man to do the same work?

15. A starts in business with b dollars; B starts with c dollars. In one year A gains as much more, while B gains $\frac{1}{2}$ as much more. How much has each at the end of the year?

16. What will 10 bushels of potatoes cost at $2m$ cents per bushel?

17. A man earns \$2 per day and pays \$ a per week for his board. How much money will he have at the end of b weeks?

18. An engine pumps 150 gallons of water into a tank each day; $10c$ gallons are drawn off each day. How much water will remain in the tank at the end of 4 days?

19. The daily wages of a mechanic are a dollars. How much will the wages of 10 mechanics for c days be?

SPECIAL CASES IN MULTIPLICATION.

58. The square of the sum of two quantities.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ ab + b^2 \\ \hline a^2 + 2ab + b^2 \end{array}$$

$$\begin{array}{r} m + n \\ m + n \\ \hline m^2 + mn \\ mn + n^2 \\ \hline m^2 + 2mn + n^2 \end{array}$$

1. How is the first term of the second power, or square, of the quantities obtained from the quantities? How is the second term obtained? The third term?

2. What signs have the terms?

59. PRINCIPLE. *The square of the sum of two quantities is equal to the square of the first quantity, plus twice the product of the first and second, plus the square of the second.*

Write out the products or powers of the following :

- | | |
|----------------------------|----------------------------|
| 1. $(x + y)(x + y)$. | 13. Square $2x + 5$. |
| 2. $(b + c)(b + c)$. | 14. Square $3m + 1$. |
| 3. $(m + 2)(m + 2)$. | 15. Square $2a + 5b$. |
| 4. $(a + x)(a + x)$. | 16. Square $a^2 + b^2$. |
| 5. $(x + 3)(x + 3)$. | 17. Square $x^2 + 3$. |
| 6. $(y + 1)(y + 1)$. | 18. Square $2m^2 + 3n^2$. |
| 7. $(2x + y)(2x + y)$. | 19. Square $ab + 2c$. |
| 8. $(m + 2n)(m + 2n)$. | 20. Square $2xy + z$. |
| 9. $(3a + b)(3a + b)$. | 21. Square $y^2 + 4z^2$. |
| 10. $(2x + 3y)(2x + 3y)$. | 22. Square $x^2 + 8$. |
| 11. $(a + 4b)(a + 4b)$. | 23. Square $5a + 7b$. |
| 12. $(2m + 2n)(2m + 2n)$. | 24. Square $4a^2 + 3b^2$. |

60. The square of the difference of the two quantities.

$x - y$	$c - d$
$x - y$	$c - d$
$x^2 - xy$	$c^2 - cd$
$- xy + y^2$	$- cd + d^2$
$x^2 - 2xy + y^2$	$c^2 - 2cd + d^2$

1. How is the first term of the second power obtained from the terms of the quantity squared? How is the second term obtained? The third term?

2. What signs connect the terms of the power?

61. PRINCIPLE. *The square of the difference of two quantities is equal to the square of the first quantity, minus twice the product of the first and second, plus the square of the second.*

Write out the products or powers of the following:

- | | |
|---------------------------|----------------------------|
| 1. $(a - x)(a - x)$. | 13. Square $2a - 3b$. |
| 2. $(b - c)(b - c)$. | 14. Square $m - 2n$. |
| 3. $(m - n)(m - n)$. | 15. Square $2b - 4d$. |
| 4. $(x - 2)(x - 2)$. | 16. Square $a^2 - 2b^2$. |
| 5. $(y - z)(y - z)$. | 17. Square $bc - xy$. |
| 6. $(a - 3b)(a - 3b)$. | 18. Square $2x^2 - 5y^2$. |
| 7. $(b - 2c)(b - 2c)$. | 19. Square $2a - c$. |
| 8. $(2x - 2y)(2x - 2y)$. | 20. Square $3m^2 - 1$. |
| 9. $(b - 5)(b - 5)$. | 21. Square $3mn - 4$. |
| 10. $(y - 1)(y - 1)$. | 22. Square $y^2 - 6$. |
| 11. $(ab - 2)(ab - 2)$. | 23. Square $4x^2 - 5y^2$. |
| 12. $(x - 4)(x - 4)$. | 24. Square $ab - 2c^2$. |

62. The product of the sum and difference of two quantities.

$x - y$	$c + d$
$x + y$	$c - d$
$x^2 - xy$	$c^2 + cd$
$xy - y^2$	$-cd - d^2$
$x^2 \qquad - y^2$	$c^2 \qquad - d^2$

1. How are the terms of the product of the sum and difference of two quantities obtained from the quantities?

2. What sign connects the terms?

63. PRINCIPLE. *The product of the sum and difference of two quantities is equal to the difference of their squares.*

Write the products of the following:

- | | |
|----------------------------|------------------------------------|
| 1. $(a + b)(a - b)$. | 13. $(b + 2c)(b - 2c)$. |
| 2. $(m + n)(m - n)$. | 14. $(3x + 8y)(3x - 8y)$. |
| 3. $(a + x)(a - x)$. | 15. $(x + 10)(x - 10)$. |
| 4. $(2a + b)(2a - b)$. | 16. $(bc + ef)(bc - ef)$. |
| 5. $(2x + y)(2x - y)$. | 17. $(3x^2 + 2y^2)(3x^2 - 2y^2)$. |
| 6. $(a + 4)(a - 4)$. | 18. $(5a + 3x)(5a - 3x)$. |
| 7. $(2m + 3n)(2m - 3n)$. | 19. $(a^2 + b^2)(a^2 - b^2)$. |
| 8. $(y + 1)(y - 1)$. | 20. $(mn + 4)(mn - 4)$. |
| 9. $(x + 5)(x - 5)$. | 21. $(x + 6)(x - 6)$. |
| 10. $(2 + y)(2 - y)$. | 22. $(4y + 7z)(4y - 7z)$. |
| 11. $(ab + 3c)(ab - 3c)$. | 23. $(3x + 4y)(3x - 4y)$. |
| 12. $(2m + 2n)(2m - 2n)$. | 24. $(2ab + 5c)(2ab - 5c)$. |

64. The product of two binomials that have a common term.

$$\begin{array}{r} x + 3 \\ x + 5 \\ \hline x^2 + 3x \\ 5x + 15 \\ \hline x^2 + 8x + 15 \end{array}$$

$$\begin{array}{r} x + 3 \\ x - 5 \\ \hline x^2 + 3x \\ -5x - 15 \\ \hline x^2 - 2x - 15 \end{array}$$

$$\begin{array}{r} x - 3 \\ x - 5 \\ \hline x^2 - 3x \\ -5x + 15 \\ \hline x^2 - 8x + 15 \end{array}$$

1. How is the first term of each product obtained from the factors?

2. How is the second term of the product in the first example obtained from the factors? The second term in the second example? The second term in the third example?

3. How is the third term of the product in each example obtained from the factors?

4. How are the signs determined which connect the terms?

65. PRINCIPLE. *The product of two binomial quantities having a common term is equal to the square of the common term, the algebraic sum of the other two multiplied by the common term, and the algebraic product of the unlike terms.*

Write the products of the following:

1. $(x + 3)(x + 4)$.

8. $(x - 4)(x + 8)$.

2. $(x - 1)(x + 5)$.

9. $(x + 9)(x + 3)$.

3. $(x - 2)(x - 3)$.

10. $(x - 12)(x + 6)$.

4. $(x + 6)(x - 7)$.

11. $(x - 5)(x - 7)$.

5. $(x + 5)(x + 10)$.

12. $(x + 14)(x - 4)$.

6. $(x - 13)(x + 3)$.

13. $(x - 1)(x + 8)$.

7. $(x + 20)(x + 5)$.

14. $(x - 5)(x - 4)$.

15. $(x + 11)(x - 2).$

20. $(x + 9)(x + 12).$

16. $(x - 25)(x - 4).$

21. $(x - 10)(x + 12).$

17. $(x + 5)(x + 15).$

22. $(x - 2y)(x + 4y).$

18. $(x - 6)(x - 3).$

23. $(x - a)(x - 7a).$

19. $(x + 6)(x - 3).$

24. $(x + 6y)(x + 10y).$

SIMULTANEOUS EQUATIONS.

66. 1. If the sum of two numbers is 8, what are the numbers? How many answers may be given to the question?

2. Let x and y stand for the two numbers; then, in the equation $x + y = 8$, how many values has x ? How many has y ? How many values has each unknown quantity in such an equation?

3. In the equation $x + y = 20$, what is the value of y , if $x = 8$? If $x = 6$? If $x = 4$? If $x = 12$? If $x = 10$?

4. If the equations $x + y = 6$ and $x - y = 2$ are added together (Ax. 2), what is the resulting equation? What is the value of x in these equations? Of y ?

67. Equations in which the same unknown quantities have the same values are called **Simultaneous Equations**.

68. The process of deducing from simultaneous equations other equations containing a less number of unknown quantities than is found in the given equations, is called **Elimination**.

69. Elimination by addition or subtraction.

1. If the equations $x + 3y = 9$ and $x - 3y = 3$ are added, what is the resulting equation? What quantity is eliminated by the addition?

2. How may the equations $2x - 4y = 4$ and $x + 4y = 8$ be combined so as to eliminate y ?

3. How may the equations $3x + 4y = 18$ and $3x + y = 9$ be combined so as to eliminate x ?

4. When may a quantity be eliminated by addition? When by subtraction?

5. If $x + 3y = 5$ and $2x + 3y = 7$, how may the value of x be found?

6. If $3x - y = 5$ and $2x + y = 5$, how may the value of x be found?

70. PRINCIPLE. *Quantities may be eliminated by addition or subtraction when they have the same coefficients.*

71. 1. Find the value of x and y in the equations $x + 3y = 11$ and $2x - 4y = 2$.

$$x + 3y = 11 \quad (1)$$

$$2x - 4y = 2 \quad (2)$$

$$4x + 12y = 44 \quad (3)$$

$$6x - 12y = 6 \quad (4)$$

$$10x = 50 \quad (5)$$

$$x = 5 \quad (6)$$

$$5 + 3y = 11 \quad (7)$$

$$3y = 6 \quad (8)$$

$$y = 2 \quad (9)$$

EXPLANATION. Since the quantities have not the same coefficients, we must multiply the equations by such numbers as will make the coefficients alike. If we wish to eliminate y , we must multiply (1) by 4 and (2) by 3 (Ax. 4). We may now eliminate y from equations (3) and (4) by addition. From the resulting equation, $10x = 50$, the value of x is obtained by dividing each member by 10, the coefficient of x .

By substituting the value of x in equation (1), equation (7) is obtained, and the value of y is found to be 2.

RULE. *If necessary, multiply one or both equations by such a quantity as will cause one unknown quantity to have the same coefficient in each equation.*

When the signs of the equal coefficients are alike, subtract one equation from the other; when the signs are unlike, add the equations.

Find the values of the unknown quantities in the following equations:

$$2. \quad \begin{cases} x + y = 6 \\ x - y = 2 \end{cases}$$

$$3. \quad \begin{cases} x - 2y = 4 \\ 2x - y = 11 \end{cases}$$

$$4. \quad \begin{cases} x + 3y = 17 \\ 2x - 2y = 2 \end{cases}$$

$$5. \quad \begin{cases} x + y = 4 \\ 4x - y = 1 \end{cases}$$

$$6. \quad \begin{cases} 5x - y = 8 \\ 2x + 2y = 20 \end{cases}$$

$$7. \quad \begin{cases} 2x - y = 3 \\ 3x + 5y = 50 \end{cases}$$

$$8. \quad \begin{cases} x + 2y = 23 \\ 3x + y = 34 \end{cases}$$

$$9. \quad \begin{cases} 4x + 2y = 14 \\ 9x - y = 4 \end{cases}$$

$$10. \quad \begin{cases} 4x - 7y = 17 \\ x + y = 7 \end{cases}$$

$$11. \quad \begin{cases} 2x - 5y = 6 \\ 5x - 12y = 16 \end{cases}$$

$$12. \quad \begin{cases} x + y = 18 \\ 7x - 2y = 9 \end{cases}$$

$$13. \quad \begin{cases} 2x + 5y = 62 \\ 6x - 5y = 2 \end{cases}$$

$$14. \quad \begin{cases} 5x - 3y = 4 \\ 5x + 3y = 16 \end{cases}$$

$$15. \quad \begin{cases} 9x + 4y = 31 \\ 2x - 8y = -2 \end{cases}$$

$$16. \quad \begin{cases} 2y + z = 26 \\ 2y + 2z = 28 \end{cases}$$

$$17. \quad \begin{cases} 2y + 3z = 23 \\ 3y - 2z = 2 \end{cases}$$

$$18. \quad \begin{cases} 8x - 10z = 10 \\ 3x + z = 18 \end{cases}$$

$$19. \quad \begin{cases} x + 2y = 30 \\ 2x + y = 18 \end{cases}$$

$$20. \quad \begin{cases} 3x - y = 30 \\ x - 3y = -30 \end{cases}$$

$$21. \quad \begin{cases} 2x + 3y = 14 \\ 3x + 2y = 16 \end{cases}$$

$$22. \quad \begin{cases} 10y - z = 17 \\ 14y - 3z = 3 \end{cases}$$

$$23. \quad \begin{cases} 5x + 2y = 6 \\ 4x + 3y = 2 \end{cases}$$

$$24. \quad \begin{cases} 6x - 3y = 12 \\ 5x - y = 13 \end{cases}$$

$$25. \quad \begin{cases} 5x - 3y = 27 \\ 3x - 2y = 16 \end{cases}$$

PROBLEMS.

72. 1. The sum of two numbers is 10, and their difference is 2. What are the numbers?

2. The sum of two numbers is 14, and the greater plus two times the less is 20. What are the numbers?

3. A and B together have \$300. Three times B's money added to five times A's gives \$1100. How much money has each?

4. The sum of the ages of a father and son is 50 years. The father's age diminished by two times the son's is 20 years. What is the age of each?

5. A boy has 25 marbles in two pockets. Twice the number in one pocket equals three times the number in the other. How many marbles has he in each pocket?

6. A farmer paid \$3400 for 100 acres of land. For part of it he paid \$30 per acre, and for part of it \$40 per acre. How many acres were bought at each price?

7. A boy spent 35 cents for oranges and pears, buying in all 13 oranges and pears. He paid 3 cents apiece for the oranges and 2 cents apiece for the pears. How many of each kind did he buy?

8. Two men start in business with \$5000 capital. Twice the amount B furnishes taken from twice the amount A furnishes will leave the amount B furnishes. What capital does each furnish?

9. There are two numbers such that five times the first minus three times the second equals 4, and two times the first plus the second equals 17. What are the numbers?

10. A farmer sold 5 horses and 7 cows to one person for \$745. To another person at the same price per head

he sold 3 horses and 10 cows for \$ 650. What was the price per head of each ?

11. A man and wife working for 6 days received \$ 15. Again, the man worked for 4 days and the wife 5 days, and they received \$ 11. What were the daily wages of each ?

12. The sum of two numbers is 26. The first minus twice the second is 8. What are the numbers ?

13. A purse contained \$ 30 in one and two dollar bills. If the whole number of bills was 18, how many bills were there of each kind ?

14. A merchant sold 2 yards of velvet and 4 yards of broadcloth for \$ 16. Again he sold 3 yards of velvet and 5 yards of broadcloth for \$ 21. What was the price of each per yard ?

15. If A gives B \$ 5 of his money, B will have twice as much money as A has left; but if B gives A \$ 5 of his money, A will have three times as much as B has left. How much money has each ?

16. A boy who desired to purchase some writing pads and some pencils found that 2 pads and 7 pencils would cost him 31 cents, and that 3 pads and 4 pencils would cost 27 cents. What was the price of each ?

17. The wages of 10 men and 8 boys per day were \$ 28, and the wages of 7 men and 10 boys at the same rate were \$ 24. What were the daily wages of each ?

18. A man received at one time \$ 17 for sawing 8 cords of wood and splitting 10 cords, and at another time \$ 13.50 for sawing 5 cords of wood and splitting 12 cords at the same rates as on the former occasion. What did he receive per cord for the sawing and for the splitting ?

DIVISION.

73. 1. Since $+5$ multiplied by $+4$ is $+20$, if $+20$ is divided by $+5$, what is the sign of the quotient?

2. What, then, is the sign of the quotient when a positive quantity is divided by a positive quantity?

3. Since $+5$ multiplied by -4 is -20 , if -20 is divided by $+5$, what is the sign of the quotient?

4. What, then, is the sign of the quotient when a negative quantity is divided by a positive quantity?

5. Since -5 multiplied by $+4$ is -20 , if -20 is divided by -5 , what is the sign of the quotient?

6. What, then, is the sign of the quotient when a negative quantity is divided by a negative quantity?

7. Since -5 multiplied by -4 is $+20$, if $+20$ is divided by -5 , what is the sign of the quotient?

8. What, then, is the sign of the quotient when a positive quantity is divided by a negative quantity?

9. What is the sign of the quotient when the dividend and the divisor have like signs? What, when they have unlike signs?

10. How many times is $6x$ contained in $12x$? $8y$ in $24y$?

11. How, then, is the coefficient of the quotient found?

12. Since $x^4 \times x^2 = x^6$, if x^6 is divided by x^2 , what is the quotient? What, when x^6 is divided by x^4 ?

13. Since $a^5 \times a^3 = a^8$, what is the quotient if a^8 is divided by a^3 ? What, if a^8 is divided by a^5 ?

14. How, then, is the exponent of a quantity in the quotient found?

74. Division is indicated in two ways:

1. By the sign $\div$, read *divided by*.

Thus, $a \div b$ shows that a is to be divided by b .

2. By writing the dividend above the divisor with a line between them.

Thus, $\frac{a}{b}$ shows that a is to be divided by b .

75. PRINCIPLES. 1. *The sign of any term of the quotient is + when the dividend and divisor have like signs, and - when they have unlike signs.*

2. *The coefficient of the quotient is equal to the coefficient of the dividend divided by that of the divisor.*

3. *The exponent of any quantity in the quotient is equal to its exponent in the dividend diminished by its exponent in the divisor.*

76. The principle relating to the signs in division may be illustrated as follows:

$$\left. \begin{array}{l} +a \times +b = +ab \\ -a \times +b = -ab \\ +a \times -b = -ab \\ -a \times -b = +ab \end{array} \right\} \text{ Hence } \left\{ \begin{array}{l} +ab \div +b = +a \\ -ab \div +b = -a \\ -ab \div -b = +a \\ +ab \div -b = -a \end{array} \right.$$

77. To divide when the divisor is a monomial.1. Divide $14x^2yz^3$ by $-7xyz$.

PROCESS.

$$\begin{array}{r|l} -7xyz & 14x^2yz^3 \\ & -2xz^2 \end{array}$$

EXPLANATION. Since the dividend and divisor have unlike signs, the sign of the quotient is — (Prin. 1.)

Then 14 divided by -7 is -2 ; x^2 divided by x (Prin. 3); y divided by y is 1, which need not appear in the quotient; z^3 divided by z is z^2 . Therefore the quotient is $-2xz^2$.2. Divide $4ax^4y^3 - 12a^2x^2y^2 - 20a^2xy^3z$ by $2axy$.

PROCESS.

$$\begin{array}{r|l} 2axy & 4ax^4y^3 - 12a^2x^2y^2 - 20a^2xy^3z \\ & 2x^3y^2 - 6axy - 10ay^2z \end{array}$$

EXPLANATION. When there are several terms in the dividend, each term must be divided separately.

RULE. Divide each term of the dividend by the divisor as follows:

Divide the coefficient of the dividend by the coefficient of the divisor. To this quotient annex each literal factor of that term of the dividend with an exponent equal to the exponent of that letter in the dividend minus its exponent in the divisor.

Write the sign $+$ before each term of the quotient when the terms of both dividend and divisor have like signs, and $-$ when they have unlike signs.

	3.	4.	5.	6.	7.	8.
Divide	$10a$	$16x$	$-14ab$	$15xy^2$	$-20a^2b^2$	$-18m^2n$
By	<u>$5a$</u>	<u>$2x$</u>	<u>$7ab$</u>	<u>$-3xy$</u>	<u>$-4ab$</u>	<u>$6m$</u>

Find the quotients in the following:

9. $30a^2bx^2 \div 15ax.$

11. $21ax^2y \div -7ay.$

10. $-24x^2y^2z^3 \div 8x^2z^2.$

12. $-9abc^2 \div -3abc.$

- | | |
|----------------------------------|---|
| 13. $30 n^2 x^2 \div 6 n^2$. | 18. $-55 abc^2 d \div 11 abc$. |
| 14. $-12 x^3 y^3 \div 12 xy$. | 19. $27 x^3 z^3 \div -9 x^2 z$. |
| 15. $-20 x^2 y^4 \div -10 y^2$. | 20. $-120 m^3 n \div -15 mn$. |
| 16. $-100 x^2 yz \div 25 xyz$. | 21. $325 x^2 y^2 z^4 \div 5 xy^2 z^2$. |
| 17. $80 x^4 y^2 \div 20 xy^2$. | 22. $-65 x^3 z^5 \div -13 x^2 z^4$. |

Divide :

23. $a^2 xy - 2 axy^2$ by ay .
24. $9 x^2 y^2 + 15 xy^2 z^2$ by $3 xy^2$.
25. $14 a^4 b^3 c + 49 a^2 bc$ by $7 abc$.
26. $-xz^3 - 3xz + x^2 z^2$ by $-xz$.
27. $4 c^2 d - 14 cd^2$ by $2 cd$.
28. $-5 x^2 y + 10 x^2 y^2 - 15 xy^2$ by $5 xy$.
29. $16 m^2 n^2 - 12 m^2 n - 8 mn^2$ by $4 mn$.
30. $15 ax^3 - 25 bx^2 y + 35 cxy^2$ by $-5 x$.
31. $9 x^2 yz - 36 xy^2 z^3 + 45 axyz^5$ by $9 xyz$.
32. $42 x^3 - 14 x^2 + 28 x + 35$ by 7 .
33. $-45 a^2 b^2 c^2 - 60 abc^2 + 30 ab^2 c^3$ by $-15 abc$.
34. $116 m^5 + 80 m^3 - 112 m^2 - 92 m$ by $4 m$.
35. $3 x^3 yz^2 - 15 x^5 y^2 z^3 + 6 x^4 yz^3 + 18 x^6 y^3 z$ by $-3 x^3 yz$.
36. $3 x^3 - 6 x^5 + 9 x^7 - 12 x^9$ by $3 x^2$.
37. $30 x^3 y^3 + 60 x^2 y^2 - 45 xy^4 + 75 x$ by $15 x$.
38. $24 abx - 16 aby + 32 a^2 bx^2 - 8 ab$ by $8 ab$.
39. $-x^5 y - x^4 yz + xy^4 z - x^3 y^3 z^2 + xy^2 z^3$ by $-xy$.
40. $50 x^3 yz^3 + 35 x^2 y^2 z - 15 ax^2 yz^3 - 20 bxy^2 z^4$ by $5 xyz$.
41. $2 n^2 x^4 y^2 - 3 nx^3 y^3 - 4 mnx^2 y^5 + 3 n^2 x^3 y^2$ by $nx^2 y^2$.
42. $a(x+y)^2 - ab(x+y)^3 + a^2 b^2(x+y)^4$ by $a(x+y)^2$.

78. To divide when the divisor is a polynomial.

1. Divide $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

PROCESS.

$$\begin{array}{r|l}
 a^3 - 3a^2b + 3ab^2 - b^3 & a - b \\
 \underline{a^3 - a^2b} & \\
 -2a^2b + 3ab^2 & \\
 \underline{-2a^2b + 2ab^2} & \\
 ab^2 - b^3 & \\
 \underline{ab^2 - b^3} &
 \end{array}$$

EXPLANATION. The divisor is written at the right of the dividend, and the quotient below the divisor. The first term of the divisor is contained in the first term of the dividend a^2 times. Therefore a^2 is the first term of the quotient. a^2 times the divisor, $a - b$, is $a^3 - a^2b$, and this subtracted from the partial dividend leaves a remainder of $-2a^2b$, to which the next term of the dividend is annexed for a new dividend.

The first term of the divisor is contained in the first term of the new dividend $-2ab$ times, consequently $-2ab$ is the second term of the quotient. $-2ab$ times the divisor, $a - b$, is $-2a^2b + 2ab^2$, and this subtracted from the second partial dividend leaves a remainder of ab^2 , to which the next term of the dividend is annexed for a new dividend.

The first term of the divisor is contained in the first term of the new dividend b^2 times, hence b^2 is the third term of the quotient. b^2 times $a - b$, is $ab^2 - b^3$, which subtracted from the third partial dividend leaves no remainder. Hence the quotient is $a^2 - 2ab + b^2$.

RULE. Write the divisor at the right of the dividend, arranging the terms of each according to the ascending or descending powers of one of the literal quantities.

Divide the first term of the dividend by the first term of the divisor, and write the result for the first term of the quotient.

Multiply the divisor by this term of the quotient, subtract the product from the dividend, and to the remainder annex one or more terms for a new dividend.

Divide the new dividend as before, and continue to divide until there is no remainder, or until the first term of the divisor is not contained in the first term of the dividend.

If there is a remainder after the last division, write it over the divisor in the form of a fraction, and annex it with its proper sign to the part of the quotient previously obtained.

2.

$$\begin{array}{r|l}
 15x^2 - 8xy - 12y^2 & \frac{3x+2y}{5x-6y} \\
 \underline{15x^2 + 10xy} & \\
 -18xy - 12y^2 & \\
 \underline{-18xy - 12y^2} &
 \end{array}$$

3.

$$\begin{array}{r|l}
 8c^2 - 26cd + 15d^2 + 8 & \frac{2c-5d}{4c-3d+\frac{8}{2c-5d}} \\
 \underline{8c^2 - 20cd} & \\
 -6cd + 15d^2 & \\
 \underline{-6cd + 15d^2} & \\
 +8 &
 \end{array}$$

4.

$$\begin{array}{r|l}
 a^2 + ab - ac - bc & \frac{a+b}{a-c} \\
 \underline{a^2 + ab} & \\
 -ac - bc & \\
 \underline{-ac - bc} &
 \end{array}$$

5.

$$\begin{array}{r|l}
 x^3 - y^3 & \frac{x-y}{x^2+xy+y^2} \\
 \underline{x^3 - x^2y} & \\
 x^2y - y^3 & \\
 \underline{x^2y - xy^2} & \\
 xy^2 - y^3 & \\
 \underline{xy^2 - y^3} &
 \end{array}$$

Divide:

6. $x^2 + 2xy + y^2$ by $x + y$.
7. $m^2 - n^2$ by $m - n$.
8. $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$.
9. $x^2 - 3x - 18$ by $x - 6$.
10. $x^2 + 8x + 15$ by $x + 3$.
11. $a^2 - 10ab - 24b^2$ by $a + 2b$.

12. $x^2 - 4x + 4$ by $x - 2$.
13. $x^4 - 49$ by $x^2 + 7$.
14. $a^6 - 16$ by $a^3 - 4$.
15. $10x^2 + 14x - 12$ by $2x + 4$.
16. $2x^3 - x^2 + 3x - 9$ by $2x - 3$.
17. $16a^2 - 24ab + 9b^2$ by $4a - 3b$.
18. $a^3 + b^3$ by $a + b$.
19. $x^3 - 8$ by $x - 2$.
20. $a^4 + a^3y + ay^3 + y^4$ by $a + y$.
21. $x^3 - 5x^2 - x + 14$ by $x^2 - 3x - 7$.
22. $x^3 - 86x - 140$ by $x - 10$.
23. $6x^3 + 4y^3 + 6xy^2 - 16x^2y$ by $3x^2 - y^2 - 2xy$.
24. $18x - 1 - 17x^2$ by $1 - x$.
25. $a^4 + 4a^2x^2 + 16x^4$ by $a^2 + 2ax + 4x^2$.
26. $27x^3 + 8y^3$ by $3x + 2y$.
27. $4x^5 - x^3 + 4x$ by $2 + 2x^2 + 3x$.
28. $x^4 - 9x^2 - 6xy - y^2$ by $x^2 + 3x + y$.
29. $36 + x^4 - 13x^2$ by $6 + x^2 + 5x$.
30. $1 - x - 3x^2 - x^5$ by $1 + 2x + x^2$.

PROBLEMS.

79. 1. Add x and y and divide the result by 2.
2. If x oranges are worth 10 cents, how much is one orange worth?
3. How many apples at 2 cents apiece, can be bought for m cents?
4. Express the quotient of b divided by a .
5. If 5 bananas cost c cents, what will 2 bananas cost?

6. How many days will it take a man to earn \$15, if he receives a dollars a day?

7. A farmer has a cows, $\frac{1}{2}$ as many horses, and three times as many sheep as horses. How many animals has he altogether?

8. Henry and James together had d marbles; Henry had c times as many as James. What represents the number that each had?

9. What is the cost per barrel, if b barrels of flour cost the sum of m and n dollars?

10. A, B, and C start in business. A furnishes 2*a* dollars, which is c times as much as B furnishes; and C furnishes b times as much as B. How much do B and C furnish?

11. A grocer mixed together equal quantities of three kinds of coffee, worth a cents, b cents, and c cents per pound. What is the cost of one pound of the mixture?

GENERAL REVIEW EXERCISES.

80. 1. Add $6(x+y)+3z-8$, $2(x+y)-2z+4$, $8z-3(x+y)$.

2. Add a^2+b^2+4ab , $5a^2-5b^2-5ab$, $-3a^2+2b^2-4ab$.

3. Add $\frac{1}{2}x + \frac{1}{3}y + z$, $x + \frac{2}{3}y + \frac{1}{3}z$, $\frac{1}{2}x + y + \frac{2}{3}z$.

4. Add $2x^m-5xy^n+6$, $3xy^n+4y^n-5x^m$, $6x^m-10+7xy^n$.

5. Add $\frac{3}{5}(m-n) + \frac{1}{5}(m+n)$, $\frac{2}{5}(m-n) - \frac{3}{5}(m+n)$, $\frac{1}{3}(m-n) - \frac{2}{5}(m+n)$.

6. From $9x+9x^2-3x^3$ subtract $2x-2x^2-12x^3$.

7. From $\frac{3}{4}a - \frac{1}{4}b + \frac{1}{6}c$ subtract $\frac{1}{2}a + \frac{1}{2}b - \frac{5}{6}c$.

8. From $5x^m y - 7x^2 y^2 + 15x^m z^n$ subtract $2x^2 y^2 - 3x^m y + 5x^m z^n$.

9. From $3(x+y)^2 - 2z^2 + 8$ subtract $2z^2 - x^2 + 4(x+y)^2$.
10. From $3a^2 - 2a^2x - 7$ subtract $7 + 3a^2 - a^2x + 2m$.
11. Multiply $m^4 - m^3 + m^2 - m + 1$ by $m + 1$.
12. Multiply $a + b + c + e$ by $a + b + c + e$.
13. Multiply $x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4$ by $x - 2y$.
14. Find the product of $x - 10$, $x + 4$, and $x + 6$.
15. Find the product of $a - b$, $a + b$, and $a^2 - b^2$.
16. Find the product of $2a - 5$, $2a - 5$, $2a - 5$, and $2a + 5$.
17. Divide $4a^4 - 5a^2b^2 + b^4$ by $2a^2 - 3ab + b^2$.
18. Divide $a^5 + 1$ by $a + 1$.
19. Divide $m^2 - c^2 + 2cz - z^2$ by $m + c - z$.
20. Divide $1 - x - 3x^2 - x^5$ by $1 + 2x + x^2$.
21. Divide $6x^4 - 96$ by $3x - 6$.

Find the value of x in the following:

22. $10x - 3x + 4 = x + 10 - 2x + 8 + x$.
23. $6x - 13 - 9x + x = 4x - 12 + 3x - 6x - 13$.
24. $5(x + 1) + 6(x + 2) = 6(x + 7)$.
25. $3(x + 1) + 4(x + 2) = 6(x + 3)$.
26. $\frac{1}{2}x - 4 + \frac{3}{4}x = 16 + \frac{1}{4}x - 10$.
27. $x(x + 5) - 6 = x(x - 1) + 12$.

Find the values of x and y in the following:

- | | |
|--|--|
| 28. $\begin{cases} 3x - 7y = 13 \\ 4x + 2y = 40 \end{cases}$ | 30. $\begin{cases} 8x - 15y = 19 \\ 2x + y = 19 \end{cases}$ |
| 29. $\begin{cases} 3x - 3y = -6 \\ 7y - 3x = 22 \end{cases}$ | 31. $\begin{cases} 4x + 2y = 24 \\ 3x - y = 8 \end{cases}$ |

32. A man being asked how much money he had, replied that \$ 25 more than three times what he had would equal \$ 775. How much money had he?

33. A man drove 155 miles in three days. On the second day he drove 15 miles farther than on the first, and on the third day he drove 20 miles farther than on the first. How far did he drive each day?

34. A and B start from two towns 231 miles apart and travel toward each other. A goes 15 miles per day, and B goes 18 miles per day. In how many days will they meet?

35. A man worked 20 days during a certain month. A part of the time he received \$ 1 per day, and part of the time \$ $1\frac{1}{2}$ per day. If he received \$ 25 for his wages, how many days did he work at \$ 1, how many at \$ $1\frac{1}{2}$ per day?

36. David and his father earned \$ 100 during a certain month. David earned \$ 10 more than $\frac{1}{2}$ as much as his father. How much did each earn?

37. A man bought an overcoat, a suit of clothes, and a pair of boots. The overcoat and the suit of clothes cost \$ 55; the overcoat and the boots cost \$ 30; and the suit of clothes and the boots cost \$ 35. What was the cost of each?

38. A and B each own half of a flock of 120 sheep. In settling the partnership, A takes 45 sheep, B takes 75 sheep and gives A \$ 45 to make the division equal. What is the value of one sheep?

39. A merchant has one quality of sugar worth 5 cents and another worth 8 cents a pound. He wishes to make a mixture of 100 pounds worth 7 cents a pound. How many pounds of each quality must he take?

FACTORING.



81. The quantities which, when multiplied together, produce a quantity, are called the **Factors** of the quantity.

Thus, a , b , and $(x + y)$ are the factors of $ab(x + y)$.

a , b , and $(x + y)$ are called the *prime factors* of $ab(x + y)$, because they have no factors besides themselves and 1.

82. A factor of a quantity is an **Exact Divisor** of it.

83. The process of separating a quantity into its factors is called **Factoring**.

84. To factor a polynomial when all the terms have a common factor.

1. What are the factors of $4a^2m - 6am^2x + 10a^2m^2x^2$?

PROCESS.

$$\begin{array}{r|l} 2am & 4a^2m - 6am^2x + 10a^2m^2x^2 \\ & 2a - 3mx + 5amx^2 \end{array}$$

EXPLANATION. By examining the terms of the polynomial, we find that $2am$ is the highest factor common to all the terms. Dividing by $2am$, we obtain the other factor. Hence, the two factors are $2am$ and $2a - 3mx + 5amx^2$.

The same result may be secured by separating the terms of the polynomial into their prime factors and then selecting the *common factors*.

RULE. Divide the polynomial by the highest factor or divisor common to all the terms. The divisor and quotient will be the *factors sought*.

Find the factors of the following polynomials :

- | | |
|--|---|
| 2. $18x^2 - 27xy$. | 12. $20m^3 - 50m^2 + 30m^4$. |
| 3. $15x^2y^2 + 20x^4y$. | 13. $xy^2 + 9x^2y^2 + 27x^3y^3$. |
| 4. $12m^2n^2 - 48mn^3$. | 14. $5am^2 + 10mn + 15mn^2$. |
| 5. $5abc - 5ac^2 + 15ab^2c$. | 15. $45x^2y^2z + 60x^2yz^2$. |
| 6. $6x^2 + 4xy - 8x^3$. | 16. $39x^2y^3 - 65x^3y + 91x^2y^3$. |
| 7. $9x^2y^2 - 6x^2y + 12x^2yz$. | 17. $32a^3b^6 + 96a^6b^8 - 8a^8b^9$. |
| 8. $6a^4b + 21a^2b - 18a^3b^3$. | 18. $25an + 75am^2 - 15an^2$. |
| 9. $3a^2b - abc - abd$. | 19. $15ax^3 - 35x^3y^3 - 55bx^3$. |
| 10. $6a^4b^5 + 36a^3b^2 - 42a^3b^3$. | 20. $45b^2x^4 + 60x^4y - 30x^4z$. |
| 11. $15a^2b^4 + 20a^3b^5 - 25a^5b^2$. | 21. $72a^3y^4 - 36a^2y^3 - 108a^2y^5$. |

85. To factor a polynomial when some of the terms have a common factor.

1. Factor $3xy + 3xz + ay + az$.

SOLUTION.

$$\begin{aligned}
 &3xy + 3xz + ay + az \\
 &= 3x(y + z) + a(y + z) \\
 &= (3x + a)(y + z)
 \end{aligned}$$

Factor the following:

- | | |
|-------------------------------------|-----------------------------------|
| 2. $ax + by + bx + ay$. | 10. $2a^4 - a^3 + 4a - 2$. |
| 3. $ab + 2b + 3a + 6$. | 11. $ax - ab - bx + b^2$. |
| 4. $9 + 3x + 3y + xy$. | 12. $x^3 + x^2 + ax + a$. |
| 5. $ax + ay - bx - by$. | 13. $ax^2 + ay^2 - bx^2 - by^2$. |
| 6. $xy + 6 - bxy - 6b$. | 14. $1 + a^2 - a^3 - a^5$. |
| 7. $y^3 - y^2 + y - 1$. | 15. $1 - x - x^2 + x^3$. |
| 8. $x^3 + x^2y - xy^2 - y^3$. | 16. $x^2y - x^2z - xy^2 + xyz$. |
| 9. $x^2y^2 + bx^2 - b^2y^2 - b^3$. | 17. $a + ay + ay^2 + ay^3$. |

86. To factor a trinomial that is a perfect square.

$$(m+n)(m+n)=m^2+2mn+n^2.$$

$$(m-n)(m-n)=m^2-2mn+n^2.$$

1. What is the product of $(m+n)(m+n)$? What, then, are the factors of $m^2+2mn+n^2$? How are the terms of the factors found from the trinomial?

2. What is the product of $(m-n)(m-n)$? What, then, are the factors of $m^2-2mn+n^2$? How are the terms of the factors found from the trinomial?

3. What term of the trinomial determines the sign which connects the terms of the binomial factors?

87. One of the *two equal* factors of a quantity is called its **square root**.

RULE. Find the square roots of the terms that are squares and connect these roots by the sign of the other term. The result will be one of the equal factors.

The other term must always be twice the product of the square roots of the terms that are squares.

Find the equal factors of the following trinomials:

1. $x^2+2xy+y^2$

9. $4x^2+8xy+4y^2$

2. x^2+4x+4

10. $x^2-10x+25$

3. $4a^2-4ab+b^2$

11. $1+2z+z^2$

4. $9m^2+6mn+n^2$

12. $a^4+4a^2b^2+4b^4$

5. $x^2-4xy+4y^2$

13. $x^2-6xz+9z^2$

6. y^2+2y+1

14. $x^2+20x+100$

7. $a^2b^2-8ab+16$

15. $m^2+8mn+16n^2$

8. $4n^2-20n+25$

16. $9x^2-18xy+9y^2$

- | | |
|---------------------------------|----------------------------------|
| 17. $9 + 6a^2 + a^4.$ | 23. $a^4 + 18a^2 + 81.$ |
| 18. $4a^4x^2 - 4a^2b^2x + b^4.$ | 24. $100x^2 - 20x + 1.$ |
| 19. $a^2m^2 - 8amn + 16n^2.$ | 25. $m^2n^2 + 4mn + 4.$ |
| 20. $25x^4 + 80x^2 + 64.$ | 26. $x^{2n} + 2x^ny^n + y^{2n}.$ |
| 21. $a^4 + 4a^2x^2 + 4x^4.$ | 27. $16 + 16m^2 + 4m^4.$ |
| 22. $4b^2c^2 - 12bcd + 9d^2.$ | 28. $1 - 8x^m + 16x^{2m}.$ |

88. To factor the difference of two squares.

$$(a + b)(a - b) = a^2 - b^2.$$

1. What is the product of $(a + b)(a - b)$? What, then, are the factors of $a^2 - b^2$? How do these two factors differ?

2. What is the product of $(x + 3)(x - 3)$? What, then, are the factors of $x^2 - 9$?

RULE. Find the square root of each term of the binomial and make the sum of these square roots one factor and their difference the other.

A binomial cannot be factored by the above rule unless the second term is *negative* and the indices of the powers are *even numbers*.

Sometimes the factors of such a quantity may themselves be resolved into factors.

$$\begin{aligned}\text{Thus,} \quad x^4 - y^4 &= (x^2 + y^2)(x^2 - y^2) \\ &= (x^2 + y^2)(x + y)(x - y).\end{aligned}$$

Factor the following:

- | | |
|--------------------|-----------------------|
| 1. $x^2 - y^2$ | 6. $x^2 - 25.$ |
| 2. $x^4 - 4.$ | 7. $121x^2 - 100y^2.$ |
| 3. $a^2 - 9b^2.$ | 8. $16a^2 - 9b^2.$ |
| 4. $m^2 - 1.$ | 9. $x^4 - 144.$ |
| 5. $a^2b^2 - c^2.$ | 10. $1 - z^2.$ |

- | | |
|-----------------------------|-------------------------|
| 11. $x^4 - y^4$. | 21. $m^4 - 625$. |
| 12. $x^2 - 16$. | 22. $225x^2 - 100y^2$. |
| 13. $9x^2 - 36$. | 23. $16y^4 - 256$. |
| 14. $4x^2 - 25$. | 24. $a^2c^2 - 9d^2$. |
| 15. $x^4 - 81$. | 25. $x^2 - 121$. |
| 16. $m^2n^2 - 64x^2$. | 26. $c^2d^2 - 1$. |
| 17. $4a^2 - 16b^2$. | 27. $25m^2 - 225n^2$. |
| 18. $x^4 - 625$. | 28. $x^6 - 1$. |
| 19. $9x^2 - 81$. | 29. $x^{2m} - y^{2n}$. |
| 20. $25a^2b^2 - 49c^4d^4$. | 30. $x^{2m} - 169$. |

89. To factor a quadratic trinomial.

$$(x + 3)(x + 2) = x^2 + 5x + 6.$$

$$(x + 3)(x - 2) = x^2 + x - 6.$$

$$(x - 3)(x + 2) = x^2 - x - 6.$$

$$(x - 3)(x - 2) = x^2 - 5x + 6.$$

1. In the above examples, what terms of the products are alike?

2. How may the first term of each factor be found from the product? How may the second term of each factor be found from the product?

3. How is the coefficient of the second term in the products found from the last terms of the factors? How, then, may the sign of the second term of each factor be found from the second term of the product?

90. A trinomial in the form $x^2 \pm ax \pm b$, in which b is the product of two quantities, and a their algebraic sum, is called a **Quadratic Trinomial**.

1. Resolve $x^2 - 9x + 18$ into two factors.

PROCESS.

$$\begin{aligned} x^2 - 9x + 18 \\ 18 = \begin{cases} 18 \times 1 \\ 9 \times 2 \\ 6 \times 3 \end{cases} \\ -9 = -6 - 3 \\ (x - 6)(x - 3) \end{aligned}$$

EXPLANATION. The first term of each factor is evidently x . Since 18 is the product of the other two quantities, and -9 is their sum, we must select from the different pairs of factors of 18 those two whose sum is -9 . Therefore, -6 and -3 are the other terms. Hence, $x - 6$ and $x - 3$ are the factors required.

2. Factor $x^2 + 4x - 21$.

SOLUTION. $21 = 3 \times 7$, or 21×1 .

The factors whose algebraic sum is $+4$ are $+7$ and -3 .

$\therefore (x + 7)$ and $(x - 3)$ are the factors.

Factor the following:

- | | |
|---------------------------|-----------------------------|
| 3. $a^2 + 6a + 8$. | 17. $x^2 - 10x - 200$. |
| 4. $x^2 + 5x - 14$. | 18. $a^2 - 6ax - 55x^2$. |
| 5. $x^2 - 6x - 7$. | 19. $x^2 - 14x + 40$. |
| 6. $m^2 - 10m + 9$. | 20. $x^2 + xy - 20y^2$. |
| 7. $b^2 + 5b + 6$. | 21. $m^2 + 10m + 24$. |
| 8. $n^2 - n - 2$. | 22. $a^2 - 4ab - 21b^2$. |
| 9. $n^2 + n - 2$. | 23. $x^2 - 14x + 48$. |
| 10. $x^2 - 7x - 18$. | 24. $x^2 + 7x - 18$. |
| 11. $x^2 + 4x - 12$. | 25. $c^2 + c - 420$. |
| 12. $a^2 - 5a - 24$. | 26. $a^2 + 11ab + 28b^2$. |
| 13. $x^2 + 5x - 36$. | 27. $x^{2n} - 9x^n + 20$. |
| 14. $y^2 - 3yz + 2z^2$. | 28. $x^4 - 3x^2 - 154$. |
| 15. $m^2 - 6mn - 16n^2$. | 29. $m^2 + 2m - 255$. |
| 16. $a^2 + 9ax + 8x^2$. | 30. $x^{2c} + 12x^c + 35$. |

91. To factor the sum or the difference of two cubes.

$$(a^3 + b^3) \div (a + b) = a^2 - ab + b^2.$$

$$(a^3 - b^3) \div (a - b) = a^2 + ab + b^2.$$

92. PRINCIPLE. *The sum of the cubes of two quantities is divisible by the sum of those quantities, and the difference of the cubes of two quantities is divisible by the difference of those quantities.*

1. Factor $a^3 - x^3$.

SOLUTION. $a^3 - x^3$ is divisible by $a - x$ (Prin.)

$$(a^3 - x^3) \div (a - x) = a^2 + ax + x^2$$

$$\therefore a^3 - x^3 = (a - x)(a^2 + ax + x^2)$$

2. Factor $r^3 + s^3$.

SOLUTION. $r^3 + s^3$ is divisible by $r + s$ (Prin.)

$$(r^3 + s^3) \div (r + s) = r^2 - rs + s^2$$

$$\therefore r^3 + s^3 = (r + s)(r^2 - rs + s^2)$$

3. Factor $a^3 - b^3c^3$.

SOLUTION. $a^3 - b^3c^3$ is divisible by $a - bc$

$$(a^3 - b^3c^3) \div (a - bc) = a^2 + abc + b^2c^2$$

$$\therefore a^3 - b^3c^3 = (a - bc)(a^2 + abc + b^2c^2)$$

Observe carefully, in the examples solved above, the quantities in the quotients and the signs, and you will be able to *write out* the factors.

Factor the following:

4. $x^3 + y^3$.

10. $s^3 + t^3$.

16. $x^3 - y^3z^3$.

5. $a^3 + b^3$.

11. $c^3 - d^3$.

17. $a^3 + b^3c^3$.

6. $m^3 - n^3$.

12. $y^3 + 1$.

18. $m^3n^3 + c^3d^3$.

7. $a^3 - b^3$.

13. $x^3 - 1$.

19. $a^3x^3 - b^3y^3$.

8. $m^3 + n^3$.

14. $a^3 - c^3d^3$.

20. $x^3 + 1$.

9. $x^3 - y^3$.

15. $m^3 + s^3$.

21. $y^3 - 1$.

EQUATIONS SOLVED BY FACTORING.

93. 1. Find the value of x in the equation $x^2 + 1 = 5$.

PROCESS.

$$x^2 + 1 = 5$$

$$x^2 = 5 - 1$$

$$x^2 = 4$$

$$x \cdot x = 2 \cdot 2$$

or $x \cdot x = -2 \cdot -2$

$$\therefore x = \pm 2$$

EXPLANATION. Transposing the known quantity to the second member, the first member contains the second power, only, of the unknown quantity. Separating each member into two equal factors,

$$x \cdot x = 2 \cdot 2 \text{ or } x \cdot x = -2 \cdot -2.$$

Hence,

$$x = +2 \text{ or } -2; \text{ or } x = \pm 2.$$

The sign, $\pm$, called the **Ambiguous Sign**, is a combination of the signs of addition and subtraction.

Thus, $a \pm b$ indicates that b may be added to or subtracted from a .

2. Find the value of x in the equation $x^2 + 5 = 30 + 11$.

SOLUTION.

$$x^2 + 5 = 30 + 11$$

$$x^2 = 30 + 11 - 5$$

$$x^2 = 36$$

$$x \cdot x = 6 \cdot 6 \text{ or } -6 \cdot -6$$

$$\therefore x = \pm 6$$

Find the value of x in the following equations:

3. $x^2 - 4 = 5$.

12. $x^2 + 21 = 25$.

4. $x^2 - 9 = 16$.

13. $x^2 + 1 = 82$.

5. $x^2 - 25 = 24$.

14. $x^2 + 12 = 48$.

6. $x^2 - 1 = 3$.

15. $x^2 - 15 = 10$.

7. $x^2 + 4 = 20$.

16. $x^2 - 10 = 90$.

8. $x^2 + 5 = 41$.

17. $x^2 + 15 - 8 = 8$.

9. $x^2 + 2 = 11$.

18. $x^2 + 3 - 6 = 5 - 2 + 10$.

10. $x^2 - 8 = 8$.

19. $x^2 - 4a^2 = 21a^2$.

11. $x^2 + 7 = 11$.

20. $x^2 + 3c^2 = 7c^2$.

94. 21. Find the value of x in the equation $x^2 + 4x + 4 = 16$.

PROCESS.

$$x^2 + 4x + 4 = 16$$

$$(x + 2)(x + 2) = 4 \cdot 4 \text{ or } -4 \cdot -4$$

$$\therefore x + 2 = 4 \text{ or } -4$$

and $x = 2 \text{ or } -6$

EXPLANATION. Since

each member of the equation can be resolved into *two equal* factors, one factor of the first member must be equal to one factor of the second member.

Hence, $x + 2 = 4$ or $x + 2$

$= -4$, and x is found to have two values, 2 or -6 .

22. Find the value of x in the equation $x^2 + 6x + 9 = 49$.

SOLUTION. $x^2 + 6x + 9 = 49$

$$(x + 3)(x + 3) = 7 \cdot 7 \text{ or } -7 \cdot -7$$

$$\therefore x + 3 = 7 \text{ or } -7$$

and $x = 4 \text{ or } -10$

Observe that the first member cannot be separated into two equal factors except when the trinomial is a perfect square (Art. 86).

Find the values of the unknown quantities in the following equations :

23. $y^2 - 10y + 25 = 16$.

32. $x^2 + 24x + 144 = 225$.

24. $x^2 - 8x + 16 = 81$.

33. $x^2 + 2x + 1 = 36$.

25. $x^2 - 16x + 64 = 9$.

34. $y^2 - 14y + 49 = 9$.

26. $z^2 + 12z + 36 = 64$.

35. $z^2 + 30z + 225 = 625$.

27. $x^2 + 14x + 49 = 100$.

36. $x^2 + 24x + 144 = 169$.

28. $x^2 - 20x + 100 = 25$.

37. $x^2 + 40x + 400 = 900$.

29. $y^2 - 18y + 81 = 16$.

38. $x^2 - 26x + 169 = 196$.

30. $x^2 + 22x + 121 = 144$.

39. $y^2 - 2y + 1 = 25$.

31. $x^2 + 6x + 9 = 4$.

40. $x^2 + 50x + 625 = 1600$.

95. 41. Find the value of x in the equation $x^2 + 4x = -4$.

PROCESS.

$$\begin{aligned} x^2 + 4x &= -4 \\ x^2 + 4x + 4 &= 0 \\ (x+2)(x+2) &= 0 \\ \therefore x+2 &= 0 \\ \text{and } x &= -2 \end{aligned}$$

EXPLANATION. By transposing -4 to the first member, that member becomes a perfect square which may be resolved into *two equal* factors, $(x+2)(x+2)$. Since the product of these factors is 0, one of the factors must be 0; and since both factors are the same, each factor is equal to 0. Hence, the value of x is -2 .

42. Find the value of x in the equation $x^2 - 8x = -16$.

SOLUTION.

$$\begin{aligned} x^2 - 8x + 16 &= 0 \\ (x-4)(x-4) &= 0 \\ \therefore x-4 &= 0 \\ \text{and } x &= 4 \end{aligned}$$

Find the values of x in the following equations:

- | | |
|-----------------------------|-----------------------------|
| 43. $x^2 + 10x + 25 = 0$. | 52. $x^2 + 20x + 100 = 0$. |
| 44. $x^2 + 12x + 36 = 0$. | 53. $x^2 + 24x = -144$. |
| 45. $x^2 + 6x + 9 = 0$. | 54. $x^2 + 36x = -324$. |
| 46. $x^2 - 18x + 81 = 0$. | 55. $x^2 - 50x = -625$. |
| 47. $x^2 + 16x + 64 = 0$. | 56. $x^2 + 22x = -121$. |
| 48. $x^2 + 26x + 169 = 0$. | 57. $x^2 - 100x = -2500$. |
| 49. $x^2 - 14x + 49 = 0$. | 58. $x^2 + 44x = -484$. |
| 50. $x^2 + 30x + 225 = 0$. | 59. $x^2 + 60x = -900$. |
| 51. $x^2 - 28x + 196 = 0$. | 60. $x^2 + 80x = -1600$. |

96. 61. Find the value of x in the equation $x^2 + 5x - 14 = 0$.

PROCESS.

$$\begin{aligned} x^2 + 5x - 14 &= 0 \\ (x+7)(x-2) &= 0 \\ \therefore x+7 &= 0 \text{ or } x-2 = 0 \\ \text{and } x &= -7 \text{ or } x = 2 \end{aligned}$$

EXPLANATION. Factoring the first member of the equation, the factors are $(x+7)(x-2)$. Since the product of the factors is equal to 0, one of the factors is equal to 0. Solving, the values of x are found to be -7 or 2 .

62. Find the value of x in the equation $x^2 + x - 72 = 0$.

SOLUTION.

$$x^2 + x - 72 = 0$$

$$(x + 9)(x - 8) = 0$$

$$\therefore x + 9 = 0 \text{ or } x - 8 = 0$$

and

$$x = -9 \text{ or } x = 8$$

Find the values of x in the following equations:

63. $x^2 - 2x - 15 = 0$.

72. $x^2 + 15x + 50 = 0$.

64. $x^2 + 6x + 5 = 0$.

73. $x^2 - x - 2 = 0$.

65. $x^2 + 10x + 9 = 0$.

74. $x^2 - 18x + 77 = 0$.

66. $x^2 - 8x + 16 = 0$.

75. $x^2 + 2x - 120 = 0$.

67. $x^2 + 2x - 48 = 0$.

76. $x^2 - 22x - 75 = 0$.

68. $x^2 + 13x + 40 = 0$.

77. $x^2 - 6ax + 8a^2 = 0$.

69. $x^2 - 5x - 24 = 0$.

78. $x^2 + 3x - 54 = 0$.

70. $x^2 + 7x + 12 = 0$.

79. $x^2 - 4ax - 96a^2 = 0$.

71. $x^2 + 9x - 22 = 0$.

80. $x^2 + 11bx + 24b^2 = 0$.

97. 81. Find the value of x in the equation $x^2 + 6x + 7 = 23$.

PROCESS.

$$x^2 + 6x + 7 = 23$$

$$x^2 + 6x + 7 + 2 = 23 + 2$$

$$x^2 + 6x + 9 = 25$$

$$(x + 3)(x + 3) = 5 \cdot 5 \text{ or } -5 \cdot -5$$

$$\therefore x + 3 = 5 \text{ or } -5$$

and

$$x = 2 \text{ or } -8$$

EXPLANATION. Equations like this may be solved in the same manner as the equations immediately preceding, by transposing all the quantities to the first member and then factoring; or they may be solved by making the first member a *perfect square* by adding to or subtracting from both members some number. The first member of

the equation is a trinomial. A trinomial is a perfect square when it is composed of two terms that are perfect squares and when the other term is twice the product of the square roots of the terms that are squares. $x^2 + 6x$ are two terms of the trinomial which is to be made a square, but the third term is to be found. Since the second term, $6x$, is twice the product of the square roots of the terms that are squares, and the square root of one of the terms is x , if $6x$ is divided by $2x$, the square root of the other term that is a square will be found. Dividing, the quotient is 3, and 3^2 , or 9, is the third term of the trinomial. Since the given term is 7, 2 must be added to both members to make the first member a perfect square, giving $x^2 + 6x + 9 = 25$. Factoring, $(x + 3)(x + 3) = 5 \cdot 5$ or $-5 \cdot -5$. Whence, $x = 2$ or -8 .

82. Find the value of x in the equation $x^2 - 12x + 33 = 46$.

SOLUTION. $x^2 - 12x + 33 = 46$

$$x^2 - 12x + 33 + 3 = 46 + 3$$

$$x^2 - 12x + 36 = 49$$

$$(x - 6)(x - 6) = 7 \cdot 7 \text{ or } -7 \cdot -7$$

$$\therefore x - 6 = 7 \text{ or } -7$$

and

$$x = 13 \text{ or } -1$$

Solve the following equations:

83. $x^2 + 10x + 20 = 11$.

92. $x^2 - 24x + 122 = 3$.

84. $x^2 + 8x + 12 = 32$.

93. $x^2 - 30x + 220 = 76$.

85. $x^2 - 18x + 80 = 15$.

94. $x^2 + 40x + 200 = 425$.

86. $x^2 + 4x + 2 = 7$.

95. $x^2 - 8x + 15 = 99$.

87. $x^2 - 20x + 85 = 10$.

96. $x^2 + 12x + 27 = 40$.

88. $x^2 + 14x + 45 = 60$.

97. $x^2 - 38x + 360 = 8$.

89. $x^2 + 22x + 100 = 60$.

98. $y^2 + 2y - 1 = 2$.

90. $x^2 + 4x + 1 = 33$.

99. $x^2 - 6x - 3 = 13$.

91. $y^2 + 16y + 54 = 90$

100. $x^2 + 8x - 2 = 18$.

COMMON DIVISORS OR FACTORS.

- 98.** 1. Name a common divisor or factor of $5x$ and $10xy$.
2. Name all the common divisors or factors of $24x^2y^2$ and $12x^3y$. Which of these is the highest common divisor or factor? Name all the common divisors or factors of $15a^2b^2$ and $25ab^4$. Which is the highest common divisor?
3. What prime factors or divisors are common to $24x^2y^2$ and $12x^3y$? To $15a^2b^2$ and $25ab^4$?
4. How may the highest common divisor or factor be obtained from the prime factors of $24x^2y^2$ and $12x^3y$? How from the prime factors of $15a^2b^2$ and $25ab^4$?

99. An exact divisor or factor of two or more quantities is called the **Common Divisor** or **Factor** of them.

100. The divisor or factor of the highest degree that is an exact divisor of two or more quantities is called the **Highest Common Divisor** or **Factor**.

The *degree* of the highest common divisor, or of any algebraic expression, is determined by the greatest number of literal prime factors in any of its terms.

Thus, $2a^2x$ and $2a^2x + 5ax$ are of the third degree, since the term $2a^2x$ has three literal prime factors, a , a , and x . Of the common divisors, a , x , and ax , ax is of the highest degree, and is, therefore, the highest common divisor.

101. PRINCIPLE. *The highest common divisor or factor of two or more quantities is equal to the product of all their common prime factors.*

102. To find the highest common divisor or factor of quantities that can be factored readily.

1. What is the highest common divisor of $4a^2b$ and $12a^2b^2c$?

PROCESS.

$$\begin{array}{r} 4a^2b = 2 \cdot 2 \cdot a \cdot a \cdot b \\ 12a^2b^2c = 3 \cdot 2 \cdot 2 \cdot a \cdot a \cdot b \cdot b \cdot c \\ \hline \text{H. C. D.} = 2 \cdot 2 \cdot a \cdot a \cdot b = 4a^2b \end{array}$$

EXPLANATION. Since the highest common divisor is the product of all the common prime factors (Prin.), the quantities are separated into their prime factors. The only prime factors common to the given quantities are 2, 2, a , a , b ; and their product is $4a^2b$. Therefore the highest common divisor is $4a^2b$.

2. What is the highest common divisor of $m(a^2 - b^2)$ and $m(a^2 - 2ab + b^2)$?

$$\begin{array}{r} \text{SOLUTION.} \quad m(a^2 - b^2) = m(a + b)(a - b) \\ m(a^2 - 2ab + b^2) = m(a - b)(a - b) \\ \hline \text{H. C. D.} = m \times (a - b) = m(a - b) \end{array}$$

Find the highest common divisor or factor of the following:

3. $14x^2yz^2$ and $21xyz^2$.
4. $18xyz^2$ and $45xyz$.
5. $5a^2b^2$, $30ab^2c$, and $15a^2bc$.
6. $22m^2n^3z$, $44m^2n^2z^2$, and $121m^2n^4z^2$.
7. $20abx^4$, $40ab^2x^3$, and $120a^2bx^2$.
8. $9a^2bm^3n^3$, $27b^4m^3n$, and $81bm^2n^2$.
9. $x^2 - 1$ and $x^2 - 2x + 1$.
10. $x^2 + 2xy + y^2$ and $x^2 - y^2$.
11. $m^3 + n^3$ and $m^2 - n^2$.

12. $a - b$, $a^2 - b^2$, and $a^2 - 2ab + b^2$.
13. $x^2 - 1$ and $x^2 - x - 2$.
14. $x^2 - 2x$ and $2xy^2 - 4y^2$.
15. $yz - z$ and $y^2 - 1$.
16. $1 - a^2$ and $1 + a^3$.
17. $x^2 + 2x - 3$ and $x^2 + 5x + 6$.
18. $x^2 - 2x - 15$ and $x^2 + 2x - 3$.
19. $x^2 - 3x - 4$ and $x^2 - x - 12$.
20. $x^2 - 1$, $x^4 - 1$, and $x^4 - 2x^2 + 1$.
21. $x^2 - 7x + 6$ and $x^2 + 3x - 4$.
22. $ax + bx$, $a^2m - b^2m$, and $a^2 + 2ab + b^2$.
23. $x^2 + 2x - 35$ and $x^2 + x - 42$.
24. $x^2 - 4xy + 4y^2$ and $x^2 - 4y^2$.
25. $x^2 - 8x + 15$, $x^2 - 4x - 5$, and $x^2 - 3x - 10$.
26. $x^2 + x - 20$, $x^2 - x - 12$, and $x^2 - 2x - 8$.
27. $x^2 + 2xy - 8y^2$ and $x^2 - 5xy + 6y^2$.
28. $x^2 + 4xy - 21y^2$ and $x^2 + 6xy - 7y^2$.
29. $x^2 + 6xy - 7y^2$ and $x^2 - 2xy + y^2$.
30. $x^2 - 4x - 5$ and $x^2 + 2x - 35$.
31. $x^2 - 4y^2$, $x^2 + 4xy - 12y^2$, and $x^2 - 4xy + 4y^2$.
32. $ax - 3a$, $x^2 - 7x + 12$, and $ax^2 + 5ax - 24a$.
33. $am + 2mx$, $a^2 + 4ax + 4x^2$, and $a^2 - 2ax - 8x^2$.
34. $b^2 - c^2$, $b^2 + 5bc + 4c^2$, and $b^2 - 9bc - 10c^2$.
35. $x^2 - 2x + 1$, $x^2 - 8x + 7$, and $x^2 - 4x + 3$.
36. $a^2 - 9$, $a^2 - 9a - 36$, and $a^2 - 7a - 30$.
37. $4a - 8y$, $a^2 - 5ay + 6y^2$, and $am - 2my$.
38. $2x + 6y$, $2(x^2 + 6xy + 9y^2)$, and $2ax + 6ay$.

COMMON MULTIPLES.

103. 1. What quantities will exactly contain 3, 5, x , and y ?

2. What quantities will exactly contain $3x^2y^2$ and $9x^3y$?

3. Give several quantities which will exactly contain $6x^2y^3z$ and $4x^2yz$. Which one of them is the lowest quantity?

4. What factors of the given quantities must be contained in the lowest quantity?

5. If any quantity is used several times as a factor in some of the given quantities, how many times will it be employed in the lowest quantity that contains the given quantities?

104. A quantity that will exactly contain a given quantity is called a **Multiple** of it.

Thus, a^3x is a multiple of a , a^2 , a^3 , x , ax , and a^2x .

105. The lowest quantity that will exactly contain each of two or more quantities is called the **Lowest Common Multiple** of the quantities.

Thus, $4x^2y$ is the lowest common multiple of $4x$, $2y$, xy , and x^2 .

106. PRINCIPLE. *The lowest common multiple of two or more quantities is equal to the product of all their different prime factors, using each factor the greatest number of times it is found in any of the given quantities.*

107. 1. What is the lowest common multiple of $12x^2yz^4$, $6a^2xy^2$, and $8axyz^2$?

PROCESS.

$$12x^2yz^4 = 4 \cdot 3 \cdot x^2 \cdot y \cdot z^4$$

$$6a^2xy^2 = 6 \cdot a^2 \cdot x \cdot y^2$$

$$8axyz^2 = 4 \cdot 2 \cdot a \cdot x \cdot y \cdot z^2$$

$$\text{L. C. M.} = 4 \cdot 3 \cdot 2 \cdot a^2 \cdot x^2 \cdot y^2 \cdot z^4 = 24a^2x^2y^2z^4.$$

EXPLANATION. By referring to the principle it is seen that the solution is dependent upon the discovery of all the different prime factors of the given quantities.

Since any number which will contain 12 will contain 6, it is not necessary to factor 6; and since 12 and 8 contain the common factor 4 and other factors, 3 and 2, which are prime numbers, 4, 3, and 2 are the numerical factors of the L. C. M. These multiplied by all the different literal factors, each with its highest exponent showing the greatest number of times it is used as a factor in any of the given quantities, will be the lowest common multiple.

2. What is the lowest common multiple of $x^2 - 3x - 40$ and $x^2 + 3x - 10$?

$$\begin{array}{rcl} \text{SOLUTION.} & x^2 - 3x - 40 & = (x - 8)(x + 5) \\ & x^2 + 3x - 10 & = (x - 2)(x + 5) \\ \hline & \text{L. C. M.} & = (x - 8)(x + 5)(x - 2) \\ & & = x^3 - 5x^2 - 34x + 80 \end{array}$$

Find the lowest common multiple of the following:

3. $8a^2b^2c$ and $18a^3bc^2$.

4. $10ab^2x$ and $15a^2bx^2$.

5. $14a^2b^2c^2$, $7b^2x^2y$, and $35abcx$.

6. $27am$, $33a^2my$, and $81a^2m^2y^2$.

7. $15x^3y^2z$ and $21x^2y^3z^2$.

8. $x^2 - 4$ and $x^2 - 4x + 4$.

9. $x^2 - y^2$ and $x^2 + 2xy + y^2$.
10. $a^2(x - y)$ and $a(x^2 - y^2)$.
11. $a^2 - b^2$, $a^2 + b^2$, and $a^4 - b^4$.
12. $x^2 - 9x - 22$ and $x^2 - 13x + 22$.
13. $x^2 + 5x + 6$ and $x^2 + 6x + 8$.
14. $x^4 - 16$, $x^2 + 4x + 4$, and $x^2 - 4$.
15. $c^2 - 5c + 4$ and $c^2 - 8c + 16$.
16. $m(a + b)$, $m^2(a - b)$, and $mx(a^2 - b^2)$.
17. $x(a^3 - b^3)$, $x(a - b)$, and $x^2(a^2 + ab + b^2)$.
18. $2a + 1$, $4a^2 - 1$, and $8a^3 + 1$.
19. $a^2 - a - 20$ and $a^2 + a - 12$.
20. $x^2y - xy^2$, $x^2 - y^2$, and $x^2 + xy$.
21. $x^2 - x$, $x^3 - 1$, $xy - y$, and $ab(x^3 - 1)$.
22. $x^4 - a^4$, $x^2 - a^2$, and $x - a$.
23. $a^2 - 5ab + 4b^2$ and $a^2 - 2ab + b^2$.
24. $x^2 - x - 30$ and $x^2 + 11x + 30$.
25. $x^2 - 1$ and $x^3 + 1$.
26. $x^2 - 1$ and $x^2 + 4x + 3$.
27. $b(a + b)$, b^2 , $am(a - b)$, and $a^2 + 2ab + b^2$.
28. $15(a^2b - ab^2)$, $21(a^3 - ab^2)$, and $35(ab^2 + b^3)$.
29. $x^2 + 5x + 6$, $x^2 - x - 12$, and $x^2 - 2x - 8$.
30. $x^3 - 8x + 15$, $x^2 - 4x - 5$, and $x^2 - 2x - 3$.
31. $x^2 + 4xy - 21y^2$, $x^2 - 2xy - 3y^2$, and $x^2 - 6xy - 7y^2$.
32. $x^2 - 3x - 28$, $x^2 + x - 12$, and $x^2 - 10x + 21$.
33. $x^2 + 2x - 35$, $x^2 + x - 42$, and $x^2 - 11x + 30$.

FRACTIONS.

108. 1. When anything is divided into two equal parts, what are the parts called? What, when it is divided into three equal parts? Into four equal parts? Into m equal parts? Into n equal parts?

2. What does $\frac{a}{3}$ represent? $\frac{c}{4}$? $\frac{x}{5}$? $\frac{n}{7}$? $\frac{1}{m}$? $\frac{1}{n}$?

3. Express $\frac{1}{5}$ of a ; $\frac{1}{3}$ of b ; $\frac{1}{8}$ of c ; $\frac{2}{3}$ of a ; $\frac{3}{5}$ of d .

109. One or more of the equal parts of anything is called a **Fraction**.

An operation in division may also be expressed as a fraction by writing the *dividend* for the *numerator* and the *divisor* for the *denominator*.

110. A quantity no part of which is in the form of a fraction is called an **Entire Quantity**.

Thus, a , $3x$, $4a + 3b$, etc., are entire quantities.

111. A quantity composed of an entire quantity and a fraction is called a **Mixed Quantity**.

Thus, $3x + \frac{2y}{5}$, $3x - 3y + \frac{a+b}{c}$ are mixed quantities.

112. The sign written before the dividing line is called the **Sign of the Fraction**.

This sign belongs to the fraction as a *whole* and not to either the numerator or the denominator.

Thus, in $-\frac{a+b}{2}$ the sign of the *fraction* is $-$, but the signs of the quantities a , b , and 2 are $+$. The sign before the dividing line simply shows whether the fraction is to be added or subtracted.

REDUCTION OF FRACTIONS.

113. To reduce fractions to higher or lower terms.

1. How many fourths are there in 1 half? In 3 halves? In 5 halves? In a halves? In b halves? In n halves?

2. How many sixths are there in 1 third? In 2 thirds? In 8 thirds? In x thirds? In y thirds? In $a + b$ thirds?

3. Since $\frac{a}{2}$ is equal to $\frac{2a}{4}$; and $\frac{x}{3}$ is equal to $\frac{2x}{6}$; what may be done to the terms of a fraction without changing the value of the fraction?

4. How many thirds are there in $\frac{4}{6}$? How many halves are there in $\frac{4}{8}$? How many fifths are there in $\frac{8}{10}$? How many twelfths are there in $\frac{6a}{24}$? In $\frac{9a}{36}$? In $\frac{16a}{48}$?

5. Change $\frac{3a}{4}$, $\frac{5a}{2}$, $\frac{6}{8a}$, $\frac{3a}{16}$ to fractions whose denominator is $16a$.

6. Reduce to equivalent fractions whose numerator is $3x$, $\frac{1}{ax}$, $\frac{3}{by}$, $\frac{3}{ab}$, $\frac{3xy}{4xy^3}$, $\frac{xyz}{4xyz}$.

7. What else, besides multiplying them by the same quantity, may be done to the terms of a fraction, without changing the value of the fraction?

114. A fraction is expressed in its **Lowest Terms** when its numerator and denominator have no *common divisor*.

115. PRINCIPLE. *Multiplying or dividing both terms of a fraction by the same quantity does not change the value of the fraction.*

116. To express a fraction in higher terms.

1. Change $\frac{2a}{5bc}$ to a fraction whose denominator is $15b^2c^2d$.

PROCESS.

$$\frac{2a}{5bc}$$

$$15b^2c^2d \div 5bc = 3bcd$$

$$\frac{2a \times 3bcd}{5bc \times 3bcd} = \frac{6abcd}{15b^2c^2d}$$

numerator must also be multiplied by $3bcd$.

EXPLANATION. Since the fraction is to be changed to an equivalent fraction expressed in higher terms, both terms of the fraction must be multiplied by the same quantity, so that the value of the fraction may not be changed (Prin.). In order to produce the required denominator, the given denominator must be multiplied by $3bcd$; consequently the

2. Change $\frac{3x}{5}$ to a fraction whose denominator is 30.

3. Change $\frac{5mn}{8}$ to a fraction whose denominator is 24.

4. Change $\frac{4a^2b}{7}$ to a fraction whose denominator is 28.

5. Change $\frac{8axy}{15b^2c}$ to a fraction whose denominator is $45b^2c^3d^2$.

6. Change $\frac{2a-5b}{11}$ to a fraction whose denominator is $33x$.

7. Change $\frac{4m-3n^2}{16}$ to a fraction whose denominator is 48.

8. Change $\frac{2xy}{3a+2}$ to a fraction whose numerator is $4x^2y$.

9. Change $\frac{6ab}{5x-3y}$ to a fraction whose numerator is $12abc$.

10. Change $\frac{4a-5b}{3}$ to a fraction whose denominator is $12xy$.

11. Change $\frac{7}{a+b}$ to a fraction whose denominator is $a^2 - b^2$.

12. Change $\frac{m+n}{m-n}$ to a fraction whose denominator is $m^2 - n^2$.

13. Change $\frac{6a-7}{15}$ to a fraction whose denominator is 75.

14. Change $\frac{1}{m+n}$ to a fraction whose numerator is $m+n$.

15. Change $\frac{2a}{x+2}$ to a fraction whose numerator is $2ax - 4a$.

16. Change $\frac{x-y}{x+y}$ to a fraction whose numerator is $x^2 - y^2$.

117. To express a fraction in its lowest terms.

1. Reduce $\frac{15x^2y^2}{20xy^2}$ to its lowest terms.

PROCESS.

$$\frac{15x^2y^2}{20xy^2} = \frac{3x}{4}$$

EXPLANATION. Since the fraction is to be changed to an equivalent fraction expressed in its lowest terms, the terms of the fraction may be divided by any quantity that will exactly divide each of them (Prin.). Dividing by the quantity, $5xy^2$, the expression is reduced to its lowest terms, since the terms have now no common factor. The terms are thus divided by their highest common divisor.

Reduce the following to their lowest terms:

2. $\frac{3abc}{9ab^2}$

5. $\frac{13m^2n}{39n^2n^2}$

8. $\frac{17m^3na^4}{68mn^2a^5}$

3. $\frac{4xy^2z}{12x^2yz^2}$

6. $\frac{112abx^2}{252abxy}$

9. $\frac{125a^5x^3z^6}{625a^7x^5z^4}$

4. $\frac{10abx}{35a^2bcx^2}$

7. $\frac{35x^3y^5z^2}{105x^2y^4z^5}$

10. $\frac{2x}{4x^2 - 6ax}$

- | | | |
|---|--|---|
| 11. $\frac{3ab}{a^2 - ab^2}$. | 17. $\frac{ax - a}{ax^2 - a}$. | 23. $\frac{x^2 - 6x - 40}{x^2 - 3x - 70}$. |
| 12. $\frac{a + b}{a^2 - b^2}$. | 18. $\frac{a^2 + a - 6}{a^2 - 9a + 14}$. | 24. $\frac{x^2 - y^2}{x^3 - y^3}$. |
| 13. $\frac{m - n}{m^2 - n^2}$. | 19. $\frac{m^2 + 8m + 15}{m^2 - 2m - 15}$. | 25. $\frac{x^2 - 1}{x^3 + 1}$. |
| 14. $\frac{a - b}{a^2 - 2ab + b^2}$. | 20. $\frac{x^2 - 2x - 24}{x^2 - 12x + 36}$. | 26. $\frac{3(x + y)}{27(x^2 - y^2)}$. |
| 15. $\frac{x^2 - y^2}{x^2 + 2xy + y^2}$. | 21. $\frac{x^2 + x - 20}{x^2 + 4x - 5}$. | 27. $\frac{2x - 3y}{8x^3 - 27y^3}$. |
| 16. $\frac{x + 1}{x^2 + 2x + 1}$. | 22. $\frac{m - n}{m^3 - n^3}$. | 28. $\frac{x^2 - 1}{2xy + 2y}$. |

118. To reduce an entire or mixed quantity to a fraction.

- How many thirds are there in 2? In 5? In 7? In a ?
- How many fifths are there in 4? In 7? In m ?
- How many fourths are there in $3\frac{1}{4}$? In $5\frac{3}{4}$? In $x + \frac{x}{4}$?

- Reduce $x + \frac{a}{y}$ to a fractional form.

PROCESS.

$$x = \frac{xy}{y}$$

$$x + \frac{a}{y} = \frac{xy}{y} + \frac{a}{y} = \frac{xy + a}{y}$$

EXPLANATION. Since 1 is equal to

$\frac{y}{y}$, x is equal to $\frac{xy}{y}$; consequently

$$x + \frac{a}{y} = \frac{xy}{y} + \frac{a}{y} \text{ or } \frac{xy + a}{y}.$$

RULE. Multiply the entire part by the denominator of the fraction; to this product add the numerator when the sign of the fraction is plus, and subtract it when it is minus, and write the result over the denominator.

If the *sign of the fraction* is $-$, the signs of all the terms in the numerator must be changed when it is subtracted.

The student must note very carefully that the *sign of the fraction* affects the *whole numerator* and not simply the first term.

Reduce the following to fractional forms :

$$2. \ 5x + \frac{3y}{2}.$$

$$14. \ x - \frac{5y - 2x}{7}.$$

$$3. \ 4x - \frac{2y}{5}.$$

$$15. \ 3x + \frac{6a - x}{ax}.$$

$$4. \ 8x + \frac{y}{4}.$$

$$16. \ a - \frac{2ac - c^2}{a}.$$

$$5. \ 2x - \frac{6y}{10}.$$

$$17. \ 1 + \frac{x}{1+x}.$$

$$6. \ x + \frac{2x - 1}{7}.$$

$$18. \ 3 + \frac{3}{x^2 - 1}.$$

$$7. \ 2x - \frac{x + 4}{3}.$$

$$19. \ 3a + \frac{ab - a}{b}.$$

$$8. \ 7x + \frac{5y - 2x}{6}.$$

$$20. \ a + x + \frac{a^2 + x^2}{a - x}.$$

$$9. \ 8a - \frac{4z + x}{5}.$$

$$21. \ a + c + \frac{2ac - c^2}{a - c}.$$

$$10. \ 3m + \frac{2m - n}{3}.$$

$$22. \ 2x - 3 - \frac{x^2 + 2}{x + 2}.$$

$$11. \ x + \frac{a - b}{c}.$$

$$23. \ m - 2n + \frac{14}{m + 2n}.$$

$$24. \ m + n - x + \frac{x^2 - 5}{m + n + x}.$$

$$12. \ 2x - \frac{y + z}{9}.$$

$$25. \ a - x + \frac{a^2 + x^2 - 5}{a + x}.$$

$$13. \ 5b + \frac{3c - 4b}{2}.$$

$$26. \ a(m + n) + \frac{an^2}{m - n}.$$

119. To reduce a fraction to an entire or mixed quantity.

1. How many units are there in $\frac{9}{8}$? In $\frac{5}{2}$? In $\frac{28}{5}$?

2. How many units are there in $\frac{3a+3b}{3}$? In $\frac{5x-10}{5}$?
In $\frac{4m+n}{2}$?

1. Reduce $\frac{ax+b}{x}$ to a mixed quantity.

PROCESS.

$$\frac{ax+b}{x} = a + \frac{b}{x}$$

EXPLANATION. Since a fraction may be regarded as an expression of unexecuted division, by performing the division indicated, the fraction is changed into the form of a mixed quantity.

Reduce the following to entire or mixed quantities:

2. $\frac{23a}{9}$

10. $\frac{x^3+1}{x-1}$

3. $\frac{26ab}{11}$

11. $\frac{2a^2-4b^2}{a+b}$

4. $\frac{45x^2y^3z}{15xy}$

12. $\frac{5ay+ax+x}{a}$

5. $\frac{36ac+4c}{9}$

13. $\frac{2ab+b^2}{a+b}$

6. $\frac{a^2+c^2}{a}$

14. $\frac{2x^2+7}{x-4}$

7. $\frac{12x^2-5y}{6x}$

15. $\frac{x^3+y^3}{x-y}$

8. $\frac{2a^2x-ax^3}{a}$

16. $\frac{2ab+ab^2-a^2}{ab}$

9. $\frac{a^2-x^2}{a-x}$

17. $\frac{x^3+2x^2-2x+1}{x^2-x-1}$

18. $\frac{2a^2 - 2b^2}{a - b}$

22. $\frac{x^2 - 4x + 6}{x - 2}$

19. $\frac{x^2 - 2x + 1}{x - 1}$

23. $\frac{2x^2 - 6x + 4}{2x - 3}$

20. $\frac{2x^2 + 5}{x - 3}$

24. $\frac{5x^2 - 7x + 5}{5x - 1}$

21. $\frac{a^2 + b^2}{a - b}$

25. $\frac{x^4 - x^3 - x + 1}{x^2 - 1}$

120. To reduce dissimilar to similar fractions.

1. Into what fractions having the same fractional unit may $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{6}$ be changed?

2. Into what fractions having the same fractional unit may $\frac{1}{3m}$ and $\frac{1}{2m}$ be changed?

3. Express $\frac{1}{3m}$, $\frac{1}{2m}$, and $\frac{1}{6m}$ in equivalent fractions having their lowest common denominator.

121. Fractions which have the same fractional unit are called **Similar Fractions**.

122. Fractions which have not the same fractional unit are called **Dissimilar Fractions**. Similar fractions have, therefore, a common denominator.

123. When similar fractions are expressed in their lowest terms, they have their **Lowest Common Denominator**.

124. PRINCIPLES. 1. *A common denominator of two or more fractions is a common multiple of their denominators.*

2. *The lowest common denominator of two or more fractions is the lowest common multiple of their denominators.*

1. Reduce $\frac{m}{2x}$ and $\frac{3}{a^2x}$ to similar fractions having their lowest common denominator.

PROCESS.

$$\frac{m}{2x} = \frac{m \times a^2}{2x \times a^2} = \frac{a^2m}{2a^2x}$$

$$\frac{3}{a^2x} = \frac{3 \times 2}{a^2x \times 2} = \frac{6}{2a^2x}$$

EXPLANATION. Since the lowest common denominator of several fractions is the lowest common multiple of their denominators (Prin. 2), the lowest common multiple of the denominators $2x$ and a^2x must be found, which is $2a^2x$. The fractions are then reduced to fractions having the denominator $2a^2x$, by multiplying the numer-

ator and denominator of each fraction by the quotient of $2a^2x$ divided by its denominator. $2a^2x \div 2x = a^2$, the multiplier of the terms of the first fraction. $2a^2x \div a^2x = 2$, the multiplier of the terms of the second fraction.

RULE. *Find the lowest common multiple of the denominators of the fractions for the lowest common denominator.*

Divide this denominator by the denominator of each fraction, and multiply the terms of the fraction by the quotient.

Reduce to similar fractions, having their lowest common denominator:

2. $\frac{a}{b}$ and $\frac{b}{c}$.

6. $\frac{2b}{x^2z}$ and $\frac{3cd}{xz^2}$.

3. $\frac{m}{n}$ and $\frac{ab}{x}$.

7. $\frac{ax}{y}$, $\frac{bc}{ax}$, $\frac{ab}{ay}$.

4. $\frac{3xy}{c}$ and $\frac{2a}{3by}$.

8. a , $\frac{bc}{xy}$, $\frac{ac}{xz}$, $\frac{ab}{yz}$.

5. $\frac{ab}{4}$ and $\frac{cd}{3y}$.

9. $\frac{2}{xy}$, $\frac{2}{ax}$, $\frac{2}{ayz}$.

$$10. \frac{m}{a^2b}, \frac{n}{bc}, \frac{mn}{b^2c^2} \qquad 14. \frac{x+y}{4}, \frac{x-y}{2c}, \frac{x^2+y^2}{2a}.$$

$$11. \frac{10a}{xy}, \frac{8b}{x^2}, \frac{9}{xy^2z} \qquad 15. \frac{5}{x+y}, \frac{7}{x-y}, \frac{x+y}{x^2-y^2}.$$

$$12. \frac{2x}{13}, \frac{2x}{3}, \frac{xy}{26}, \frac{4y}{39} \qquad 16. \frac{3}{a+b}, \frac{4}{a-b}.$$

$$13. \frac{a+b}{3x}, \frac{a-b}{2y}, \frac{ab}{6} \qquad 17. \frac{3}{x+2}, \frac{5}{x-2}, \frac{9}{x}.$$

$$18. \frac{a+b}{a^3-b^3}, \frac{1}{a-b}, \frac{1}{a^2+ab+b^2}.$$

$$19. \frac{1}{x+1}, \frac{3}{4x+4}, \frac{x}{x^2-1}.$$

$$20. \frac{2}{a^2-b^2}, \frac{1}{a-b}, \frac{3}{a^2+b^2}.$$

$$21. \frac{x+2}{x^2-3x+2}, \frac{x-2}{x^2+x-2}.$$

$$22. \frac{x}{x^2+2x-3}, \frac{a}{x^2-2x-15}.$$

$$23. \frac{x+1}{x(x-2)}, \frac{x-1}{4x-8}, \frac{2}{4x}.$$

$$24. \frac{5}{1+2x}, \frac{3x}{1-2x}, \frac{4-13x}{1-4x^2}.$$

$$25. \frac{1}{a+b}, \frac{a}{a^2-b^2}, \frac{3a}{a^4-b^4}.$$

$$26. \frac{x-y}{x^2-xy+y^2}, \frac{x+y}{x^3+y^3}, \frac{x^2}{xy+y^2}.$$

CLEARING EQUATIONS OF FRACTIONS.

125. 1. Five is one half of what number?

2. Eight is one third of what number? One fifth of what number?

3. If $\frac{1}{2}x$ equals 5, what is the value of x ?

4. If $\frac{1}{3}x = 6$, what is the value of x ?

5. What effect has it upon the equality of the members of an equation to multiply both by the same quantity?

6. If the members of the equation $\frac{1}{5}x = 6$ are multiplied by 5, what is the resulting equation?

7. Multiply the following equations by such quantities as will change them into equations without fractions:

$$\frac{1}{3}x = 8, \text{ or } \frac{x}{3} = 8; \quad \frac{1}{6}x = 7, \text{ or } \frac{x}{6} = 7;$$

$$\frac{x}{8} = 4; \quad \frac{x}{10} = 9; \quad \frac{x}{2} + \frac{x}{4} = 6; \quad \frac{x}{3} + \frac{x}{6} = 5;$$

$$\frac{x}{4} + \frac{x}{8} = 10; \quad \frac{x}{5} + \frac{x}{10} = 20; \quad \frac{x}{3} + \frac{x}{5} = 8.$$

8. How may an equation containing fractions be changed into an equation without fractions?

126. Changing an equation containing fractions into another equation without fractions is called **Clearing an Equation of Fractions**.

127. PRINCIPLE. *An equation may be cleared of fractions by multiplying both members by some common multiple of the denominators of the fractions.*

1. Find the value of x in the equation $x + \frac{x}{3} = 8$.

PROCESS.

$$x + \frac{x}{3} = 8$$

Clearing of fractions, $3x + x = 24$

Uniting terms, $4x = 24$

Therefore, $x = 6$

EXPLANATION. Since the equation contains a fraction, it may be cleared of fractions by multiplying each member by the denominator of the fraction (Prin.). The denominator is 3; therefore each member is multiplied by 3; thus, 3 times x is $3x$, 3 times $\frac{x}{3}$, or $\frac{1}{3}x$, is x , or $\frac{1}{3}x$, and 3 times 8 is 24. The resulting equation is $3x + x = 24$; therefore $x = 6$.

2. Find the value of x in the equation

$$x + \frac{x}{4} + \frac{x}{2} + \frac{x}{5} = \frac{39}{2}$$

PROCESS.

$$x + \frac{x}{4} + \frac{x}{2} + \frac{x}{5} = \frac{39}{2}$$

Clearing of fractions, $20x + 5x + 10x + 4x = 390$

Therefore, $39x = 390$

and $x = 10$

EXPLANATION. Since the equation may be cleared of fractions by multiplying by some multiple of the denominators (Prin.), this equation may be cleared of fractions by multiplying both members by 4, 2, 5, and 2 successively, or by their product, or by any multiple of 4, 2, 5, and 2.

Since the multiplier will be the smallest when we multiply by the lowest common multiple of the denominators, we multiply both members by 20, the lowest common multiple of 4, 2, 5, and 2. Uniting terms and dividing by the coefficient of x , the result is $x = 10$.

RULE. *Multiply both members of the equation by the least, or lowest, common multiple of the denominators.*

1. An equation may also be cleared of fractions by multiplying each member by the product of all the denominators.

2. Multiplying a fraction by its denominator removes the denominator.

3. If a fraction has the *minus* sign before it, *the signs of all the terms of the numerator must be changed when the denominator is removed.*

3. Find the value of x in the equation

$$\frac{x-3}{4} - \frac{x-1}{9} = \frac{x-5}{6}.$$

SOLUTION.

$$\frac{x-3}{4} - \frac{x-1}{9} = \frac{x-5}{6}$$

Clearing of fractions,

$$9x - 27 - 4x + 4 = 6x - 30$$

Transposing,

$$9x - 4x - 6x = 27 - 4 - 30$$

Uniting terms,

$$-x = -7$$

Dividing by -1 ,

$$x = 7$$

Note 3 under the rule is a very important one. Observe its application in the above solution.

Find the value of x and verify the result in the following :

4. $x + \frac{x}{4} = 10.$

10. $x + \frac{x}{2} + \frac{x}{3} = 11.$

5. $x + \frac{x}{3} = 20.$

11. $\frac{x}{3} + \frac{x}{4} + \frac{x}{6} = 9.$

6. $2x + \frac{x}{4} = 9.$

12. $2x + \frac{x}{2} + \frac{x}{7} = 37.$

7. $2x + \frac{x}{8} = 34.$

13. $x + \frac{x}{3} + \frac{x}{5} = \frac{69}{5}.$

8. $4x + \frac{x}{4} = 51.$

14. $x + \frac{x}{5} + \frac{x}{10} + \frac{x}{15} = 41.$

9. $\frac{x}{3} + 3x = 10.$

15. $\frac{x}{3} + \frac{x}{6} + \frac{x}{8} + \frac{x}{12} = 17.$

$$16. 3x + \frac{x}{9} + \frac{x}{12} = \frac{115}{12}.$$

$$24. \frac{x}{10} - \frac{2x}{5} + \frac{3x}{2} - \frac{x}{4} = \frac{19}{4}.$$

$$17. 2x + \frac{2x}{4} + \frac{3x}{15} = 27.$$

$$25. \frac{x}{3} + \frac{x+2}{8} = x-3.$$

$$18. 3x - \frac{x}{6} + \frac{x}{12} = 70.$$

$$26. \frac{x}{3} - \frac{x-1}{11} = x-9.$$

$$19. x + \frac{x}{7} - \frac{x}{5} = 33.$$

$$27. \frac{9x}{7} - 2x = \frac{x+3}{5} - 21.$$

$$20. x + \frac{3x}{5} + \frac{2x}{6} - \frac{x}{2} = \frac{43}{2}.$$

$$28. \frac{3x+4}{3} = \frac{40}{3} - 2x.$$

$$21. \frac{2x}{3} - \frac{x}{5} + \frac{x}{4} + \frac{x}{12} = 8.$$

$$29. 2 - \frac{x+8}{4} = 2x - \frac{x+6}{3}.$$

$$22. 2x + \frac{x}{8} - \frac{2x}{3} + \frac{x}{6} = \frac{13}{2}.$$

$$30. \frac{x+3}{2} = \frac{2x+3}{10} + \frac{7x-5}{5}.$$

$$23. x + \frac{x}{2} - \frac{x}{4} + \frac{x}{11} = 59.$$

$$31. \frac{7x+16}{21} = \frac{x+8}{21} + \frac{x}{3}.$$

$$32. \frac{x+3}{4} - \frac{x-8}{5} = \frac{x-5}{2} - 1.$$

$$33. \frac{6x-14}{8} + \frac{2x-1}{3} = 2x-5.$$

$$34. \frac{23x+13}{4} - \frac{x-1}{2} = 6.$$

$$35. \frac{4x+2}{11} = \frac{3x-5}{13} + 1.$$

$$36. \frac{x+1}{3} + \frac{x+3}{4} = \frac{x+4}{5} + 16.$$

$$37. \frac{x-7}{10} + \frac{x-7}{5} = \frac{x+1}{6}.$$

$$38. \frac{x+3}{2} - \frac{x-1}{3} = \frac{x+42}{7} - \frac{x+5}{3}.$$

$$39. \quad \frac{x}{2} + x = \frac{x}{3} + \frac{6x+1}{6}. \quad 42. \quad \frac{x}{5} + \frac{x-2}{3} = x - \frac{5x-1}{6}.$$

$$40. \quad \frac{x}{3} - \frac{2x-4}{7} = x - \frac{x+7}{2}. \quad 43. \quad \frac{x}{8} + \frac{x}{5} - \frac{2x}{5} = x - \frac{52}{4}.$$

$$41. \quad \frac{1-2x}{3} = \frac{4-5x}{4} - \frac{13}{42}. \quad 44. \quad \frac{x-4}{4} - \frac{x-1}{3} = \frac{x-26}{5}.$$

$$45. \quad \frac{x-1}{2} + \frac{x-3}{4} - \frac{x-2}{3} = \frac{2}{3}.$$

PROBLEMS.

128. 1. A certain number diminished by $\frac{1}{5}$ and also by $\frac{1}{6}$ of itself leaves a remainder of 19. What is the number?

SOLUTION. Let

x = the number.

Then

$\frac{x}{5}$ and $\frac{x}{6}$ = the parts of the number.

And

$$x - \frac{x}{5} - \frac{x}{6} = 19$$

Clearing of fractions, $30x - 6x - 5x = 570$

Uniting,

$$19x = 570$$

$$x = 30$$

2. What number is that the sum of whose third and fourth parts equals its sixth part plus 5?

3. A man left $\frac{1}{3}$ of his property to his wife, $\frac{1}{2}$ to his children, and the remainder, which was \$1200 to a public library. What was the value of his property?

4. Out of a cask of wine $\frac{1}{5}$ part leaked away; afterward 10 gallons were drawn out. The cask was then $\frac{2}{3}$ full. How many gallons did it hold?

5. A man leased some property for 40 years. $\frac{1}{2}$ of the time the lease has run is equal to $\frac{1}{3}$ of the time it has to run. How many years has it run?

6. The sum of two numbers is 35, and $\frac{1}{3}$ of the less is equal to $\frac{1}{4}$ of the greater. What are the numbers?

7. A man paid $\frac{2}{3}$ as much for a wagon as for a horse, and the price of the horse plus $\frac{1}{2}$ of the price of the wagon was 100 dollars. What was the price of each?

8. The sum of two numbers is 76, and $\frac{2}{5}$ of the less is equal to $\frac{1}{7}$ of the greater. What are the numbers?

9. There is a certain number the sum of whose fifth and seventh parts exceeds the difference of its seventh and fourth parts by 99? What is the number?

10. Divide the number 50 into two such parts that $\frac{1}{2}$ of one plus $\frac{3}{4}$ of the other shall equal 35.

11. A son's age is $\frac{2}{5}$ of his father's; but 15 years ago the son's age was $\frac{1}{7}$ of the father's. What are the ages of each?

12. A man paid \$3 a head for some sheep. After 20 of them had died, he sold $\frac{1}{4}$ of the remainder at cost for \$60. How many sheep did he buy?

13. A man wished to distribute some money among a certain number of children. He found that, if he gave to each of them 8 cents, he would have 10 cents left, but, if he gave to each 10 cents, he lacked 10 cents of having enough. How many children were there, and how much money had the man?

14. A man and his oldest son can earn \$30 a week; the man and his youngest son can earn $\frac{5}{6}$ as much; and the two sons can earn \$19 a week. How much can each alone earn per week?

15. A and B start in business with the same capital. A gains \$1775, and B loses \$225. B then has $\frac{2}{3}$ as much money as A. What was their original capital?

16. Divide \$440 between A, B, and C so that B shall have \$5 more than A, and C shall have $\frac{3}{5}$ as much as A and B.

ADDITION AND SUBTRACTION OF FRACTIONS.

129. 1. Find the value of $\frac{1}{2} + \frac{1}{6}$; of $\frac{1}{2} - \frac{1}{4}$; of $\frac{2a}{3} + \frac{a}{6}$.

2. What kind of fractions can be added or subtracted without changing their form?

3. What must be done to dissimilar fractions before they can be added or subtracted? How are dissimilar fractions made similar?

130. PRINCIPLES. 1. *Only similar fractions can be united into a single term by addition and subtraction.*

2. *Dissimilar fractions must be reduced to similar fractions before they can be united by addition or subtraction into one term.*

1. What is the sum of $\frac{a}{2}$, $\frac{2a}{5}$, and $\frac{3c}{4}$?

$$\text{SOLUTION. } \frac{a}{2} + \frac{2a}{5} + \frac{3c}{4} = \frac{10a}{20} + \frac{8a}{20} + \frac{15c}{20} = \frac{18a + 15c}{20}.$$

2. Subtract $\frac{2a}{3x}$ from $\frac{7b}{8y}$.

$$\text{SOLUTION. } \frac{7b}{8y} - \frac{2a}{3x} = \frac{21bx}{24xy} - \frac{16ay}{24xy} = \frac{21bx - 16ay}{24xy}.$$

3. Simplify $\frac{x+2}{x-2} + \frac{x-2}{x+2} - \frac{x^2+4}{x^2-4}$.

SOLUTION. The lowest common denominator is $x^2 - 4$,

$$\begin{aligned} & \frac{x+2}{x-2} + \frac{x-2}{x+2} - \frac{x^2+4}{x^2-4} \\ &= \frac{x^2+4x+4}{x^2-4} + \frac{x^2-4x+4}{x^2-4} - \frac{x^2+4}{x^2-4} = \frac{x^2+4}{x^2-4}. \end{aligned}$$

Add:

$$4. \frac{2}{ab} \text{ and } \frac{3}{ab}.$$

$$5. \frac{2ab}{3xy} \text{ and } \frac{5ad}{3xy}.$$

$$6. \frac{x}{y} \text{ and } \frac{m}{n}.$$

$$7. \frac{x}{3a} \text{ and } \frac{3y}{6ax}.$$

$$8. \frac{a}{a+x} \text{ and } \frac{a}{a-x}.$$

$$9. \frac{a+b}{4} \text{ and } \frac{a^2}{a-b}.$$

$$10. \frac{2a+b}{3c} \text{ and } \frac{5a-4b}{12c}.$$

$$11. \frac{2x^2}{x^2-y^2} \text{ and } \frac{2x}{x+y}.$$

$$12. \frac{a+4}{a+3} \text{ and } \frac{a+2}{a+5}.$$

$$13. \frac{2x-3y}{2x-y} \text{ and } \frac{2x-y}{x-y}.$$

$$14. \frac{5}{m+n} \text{ and } \frac{7}{m-n}.$$

$$15. \frac{x+1}{x-1} \text{ and } \frac{x-1}{x+1}.$$

$$16. \frac{y^2-2xy-x^2}{x^2-xy} \text{ and } \frac{x}{x-y}.$$

$$17. \frac{-3xy}{x^3-y^3} \text{ and } \frac{3}{x-y}.$$

$$18. \frac{a-7}{a+2} \text{ and } \frac{a+4}{a-5}.$$

$$19. \frac{a-b}{a+b} \text{ and } \frac{a+b}{a-b}.$$

$$20. \frac{5}{2x-1} \text{ and } \frac{2x-7}{4x^2-1}.$$

$$21. \frac{1}{x+y} \text{ and } \frac{2y}{x^2-y^2}.$$

$$22. \frac{a}{x(a-x)} \text{ and } \frac{x}{a(a-x)}.$$

$$23. \frac{3a}{a+x} \text{ and } \frac{2ax}{a^2-x^2}.$$

Remember to *change the signs of all the terms in the numerator* when you subtract a fraction.

Subtract:

$$24. \frac{3x}{9} \text{ from } \frac{5x}{9}.$$

$$25. \frac{b}{5} \text{ from } \frac{2a}{5}.$$

$$26. \frac{mn}{ax} \text{ from } \frac{bc}{y}.$$

$$27. \frac{2b}{xy} \text{ from } \frac{cd}{yz}.$$

$$28. \frac{1}{x-1} \text{ from } \frac{1}{x-2}.$$

$$31. \frac{3}{x+2} \text{ from } \frac{x}{x+3}.$$

$$29. \frac{2x-3}{x^2-4} \text{ from } \frac{5}{x-2}.$$

$$32. \frac{5}{x^2-2x} \text{ from } \frac{6}{2x}.$$

$$30. \frac{a-b}{a^2-b^2} \text{ from } \frac{a+b}{a^2-2ab+b^2}.$$

$$33. \frac{1}{(x-a)^2} \text{ from } \frac{1}{(x+a)^2}.$$

34. Find the sum of one half of c dollars and one third of c plus d dollars.

35. A man having b dollars paid out one half of his money, and then one fifth of the remainder. How much did he pay out? How much had he left?

36. A carpet cost a dollars, a table b dollars, and a desk one half as much as the carpet and the table. What was the cost of all?

Simplify:

$$37. \frac{2xy}{15} + \frac{4xy}{5} - \frac{3xy}{7}.$$

$$44. \frac{a-b}{3} + \frac{a+b}{4} - \frac{a}{2} + \frac{b}{5}.$$

$$38. \frac{2}{3}x - \frac{1}{6}x + \frac{1}{4}x.$$

$$45. \frac{x}{y} + \frac{y}{x+y} + \frac{x^2}{x^2+xy}.$$

$$39. \frac{5}{11}ab - \frac{1}{2}ab + \frac{1}{3}ab.$$

$$46. \frac{xy}{x+y} - \frac{xy}{x-y} + \frac{1}{x-y}.$$

$$40. \frac{m}{3} + \frac{2m}{4} - \frac{5n}{6}.$$

$$47. \frac{1}{1+a} + \frac{1}{1-a} + \frac{2a}{1-a^2}.$$

$$41. \frac{3x}{5} + \frac{x}{4} + \frac{3x}{10}.$$

$$48. \frac{a-b}{ab} + \frac{b-c}{bc} - \frac{c-a}{ac}.$$

$$42. \frac{2a}{7} - \frac{4b}{9} + \frac{a}{3}.$$

$$49. \frac{2x}{x^2-1} - \frac{3}{x+1} + \frac{5}{x-1}.$$

$$43. \frac{5ax}{6} - \frac{2ax}{9} + \frac{5ax}{12}.$$

$$50. \frac{2x-5}{x^2-x-2} - \frac{3}{x-2} + \frac{7}{x+1}.$$

$$53. x - \frac{x^2}{x+1} + \frac{x}{x-1}.$$

$$51. \frac{1}{x+1} + \frac{2}{x+2} + \frac{3}{x-3}.$$

$$54. \frac{1}{x-a} + \frac{1}{x+a} - \frac{2}{x}.$$

$$52. \frac{1}{x^2-x-2} - \frac{1}{x^2+x-2}.$$

$$55. \frac{a}{a-x} + \frac{3a}{a+x} - \frac{2ax}{a^2+x^2}.$$

56. A man walked a miles the first day; the second day he walked half as far as the first day; and the third day half as far as the second. How far did he walk in the 3 days?

57. What is the cost of a coat, hat, and pair of shoes, if the coat costs a dollars, the hat c dollars, and the shoes $\frac{2}{3}$ as much as the hat and coat?

58. A train went e miles the first hour, f miles the second, and then went back $\frac{1}{3}$ of the whole distance. How far was the train then from the starting point?

MULTIPLICATION OF FRACTIONS.

131. 1. How much is three times $\frac{2}{7}$? Four times $\frac{3}{5}$? Five times $\frac{2a}{9}$?

2. Express $2 \times \frac{3}{4}$ in its lowest terms; $5 \times \frac{7}{15}$; $8 \times \frac{5}{24}$; $3 \times \frac{2a}{9}$.

3. How may the products in 1 and 2 be obtained from the given fractions?

4. In what two ways, then, may a fraction be multiplied by an integer?

5. How much is $\frac{1}{3}$ of $\frac{2}{5}$ or $\frac{2}{5} \div 3$? $\frac{1}{2}$ of $\frac{4a}{5}$ or $\frac{4a}{5} \div 2$?

6. In what two ways, then, may a fraction be divided by an integer?

132. PRINCIPLES. 1. *Multiplying the numerator or dividing the denominator of a fraction by any quantity, multiplies the fraction by that quantity*

2 *Dividing the numerator or multiplying the denominator of a fraction by any quantity, divides the fraction by that quantity.*

1. What is the product of $\frac{x}{y}$ multiplied by $\frac{c}{3}$?

PROCESS.	EXPLANATION. To multiply $\frac{x}{y}$ by $\frac{c}{3}$ is to find c
$\frac{x}{y} \times \frac{c}{3} = \frac{cx}{3y}$	times $\frac{1}{3}$ part of $\frac{x}{y}$. $\frac{1}{3}$ part of $\frac{x}{y}$ is $\frac{x}{3y}$ (Prin. 2), and c
	times $\frac{x}{3y}$ is $\frac{cx}{3y}$ (Prin. 1). That is, <i>the product of the</i>
	<i>numerators is the numerator of the product, and the product of the</i>
	<i>denominators is the denominator of the product.</i>

1. Reduce all entire and mixed quantities to the fractional form before multiplying.

2. Entire quantities may be expressed in the form of a fraction by writing 1 as a denominator. Thus, a may be written $\frac{a}{1}$.

3. When possible, factor the terms of the fractions and *cancel* equal factors from numerator and denominator.

2. Find the product of $\frac{x-3}{x^2-4} \times \frac{x^2-1}{4x} \times \frac{x+2}{x^2-2x-3}$.

SOLUTION. $\frac{x-3}{x^2-4} \times \frac{x^2-1}{4x} \times \frac{x+2}{x^2-2x-3}$

$$= \frac{x-3}{(x+2)(x-2)} \times \frac{(x+1)(x-1)}{4x} \times \frac{x+2}{(x-3)(x+1)}$$

Canceling equal factors from dividend and divisor, the result is

$$\frac{x-1}{4x(x-2)}.$$

Multiply:

$$3. \frac{m^2 n}{abcx} \text{ by } abn.$$

$$9. \frac{m-n}{18} \text{ by } \frac{6}{m+n}.$$

$$4. \frac{5c^2 d}{27ax^2} \text{ by } 6a.$$

$$10. \frac{11m^2 n^2}{x^2 - y^2} \text{ by } x + y.$$

$$5. \frac{3a}{4b} \text{ by } \frac{d}{a}.$$

$$11. \frac{2a}{x+y} \text{ by } \frac{5ab}{x-y}.$$

$$6. \frac{2m}{5ax} \text{ by } \frac{ab}{xyz}.$$

$$12. \frac{3ab}{x^2 - 4} \text{ by } \frac{x-2}{a}.$$

$$7. \frac{7am}{9bn} \text{ by } \frac{3mn}{2ax}.$$

$$13. \frac{a-x}{x^2} \text{ by } \frac{a^2}{a^2 - x^2}.$$

$$8. \frac{x+y}{abc} \text{ by } \frac{bc}{ax}.$$

$$14. \frac{x+4}{5x-15} \text{ by } \frac{x-3}{x^2+5x+4}.$$

$$15. \frac{a^2 - ab}{a^2 - 2ab + b^2} \text{ by } \frac{a^2 - 4ab + 3b^2}{a^2 - b^2}.$$

$$16. \frac{x^2 - x - 2}{x^2 + 4x + 3} \text{ by } \frac{x^2 - 3x - 10}{x^2 - 4}.$$

$$17. \frac{3m - 12}{m^2 + 3m + 2} \text{ by } \frac{m + 2}{m^2 - 5m + 4}.$$

$$18. \frac{x+3}{5x-15} \text{ by } \frac{x^2 - 9}{x^2 - 4x - 21}.$$

$$19. \frac{20}{x^2 - 64} \text{ by } \frac{x^2 + 3x - 40}{5x - 25}.$$

$$20. \frac{a^4 - b^4}{a^2 b^2} \text{ by } \frac{ab^3}{a^2 + b^2}.$$

$$21. \frac{x^2 + 2x - 63}{x^2 - 4x - 21} \text{ by } \frac{x^2 + 9x + 18}{6x + 54}.$$

Simplify the following :

$$22. \frac{a}{bc} \times \frac{c^2}{ab} \times \frac{b^2}{ac}.$$

$$23. \frac{x^2 - y^2}{x} \times \frac{x}{x + y} \times \frac{a}{x - y}$$

$$24. \left(2 - \frac{x^2}{8}\right) \left(\frac{4}{x + 4}\right).$$

$$25. \left(\frac{1}{x} + \frac{1}{y}\right) \left(\frac{x^2 y}{x^2 - y^2}\right).$$

$$26. \left(\frac{x + 5}{x^2 + 10x + 25}\right) \left(\frac{x^2 - 25}{x - 2}\right) \left(\frac{x^2 - 4}{x - 5}\right).$$

$$27. \left(\frac{x}{2} + \frac{x}{3} - \frac{2y}{4}\right) \left(\frac{3xy}{5x^2 - 3xy}\right).$$

$$28. \frac{4x^2 y}{9xz} \times \frac{2ax}{x^2 - y^2} \times \frac{3(x + y)}{8xy^2} \times \frac{x - y}{xyz}.$$

$$29. \left(\frac{1}{m^2 + m + 1}\right) \left(\frac{1}{m^2} + \frac{1}{m} + 1\right).$$

$$30. \frac{a + 1}{a + 3} \times \frac{a - 2}{a - 5} \times \frac{a^2 - 2a - 15}{a^2 - a - 2}.$$

31. If a cords of wood cost b dollars, how much will c cords cost at the same rate ?

32. If b pounds of sugar are worth c pounds of butter, how many pounds of butter are 6 pounds of sugar worth ?

33. A father works a weeks at n dollars per week, and his son works 4 weeks at $\frac{1}{3}$ of the wages that the father receives. How much do both receive ?

34. What is the value of d acres of land at $\frac{2}{3}$ of $(d - e)$ dollars per acre ?

35. A walked c miles in b hours. How far did he walk in 1 hour ? In 5 hours ? In a hours ? In $c + d$ hours ?

DIVISION OF FRACTIONS.

133. 1. How many times is $\frac{1}{5}$ contained in 1? $\frac{1}{6}$ in 1?
 $\frac{1}{12}$ in 1? $\frac{1}{15}$ in 1? $\frac{1}{a}$ in 1? $\frac{1}{3b}$ in 1? $\frac{1}{m+n}$ in 1?

2. How does the number of times a fraction, having 1 for its numerator, is contained in 1 compare with its denominator?

3. How many times is $\frac{3}{4}$ contained in 1? $\frac{5}{8}$ in 1?
 $\frac{5}{c}$ in 1? $\frac{2}{d}$ in 1?

1. What is the value of $\frac{b}{2c} \div \frac{a}{d}$?

PROCESS.

$$\frac{b}{2c} \div \frac{a}{d} = \frac{b}{2c} \times \frac{d}{a} = \frac{bd}{2ac}$$

since $\frac{a}{d}$ is contained in 1 $\frac{d}{a}$ times, it will be contained in $\frac{b}{2c}$, $\frac{b}{2c}$ times $\frac{d}{a}$ or $\frac{bd}{2ac}$ times. That is, *the quotient of one fraction divided by another is equal to the product of the dividend by the divisor inverted.*

EXPLANATION. The fraction $\frac{1}{d}$ is con-

tained in 1 d times; and $\frac{a}{d}$ is contained in 1, $\frac{1}{a}$ part of d times, or $\frac{d}{a}$ times. And,

since $\frac{a}{d}$ is contained in 1 $\frac{d}{a}$ times, it will be contained in $\frac{b}{2c}$, $\frac{b}{2c}$ times $\frac{d}{a}$ or $\frac{bd}{2ac}$ times. That is, *the quotient of one fraction divided by another is equal to the product of the dividend by the divisor inverted.*

1. Change entire and mixed quantities to the fractional form.

2. An entire quantity may be expressed as a fraction by writing 1 for its denominator.

3. When possible, factor the terms of the fractions and use cancellation.

2. Divide $\frac{x^2-1}{x^2+x-20}$ by $\frac{x+1}{x^2-25}$.

$$\text{SOLUTION. } \frac{x^2-1}{x^2+x-20} \div \frac{x+1}{x^2-25} = \frac{(x+1)(x-1)}{(x+5)(x-4)} \times \frac{(x+5)(x-5)}{x+1}$$

Canceling equal factors, the quotient is

$$\frac{(x-1)(x-5)}{x-4} \text{ or } \frac{x^2-6x+5}{x-4}$$

Divide:

3. $\frac{x}{y}$ by $\frac{m}{n}$.
4. $\frac{ab}{xy}$ by $\frac{b}{c}$.
5. $\frac{x^2y}{z}$ by $\frac{xy}{z^2}$.
6. $\frac{15ab}{6x^2}$ by $\frac{5x}{3b}$.
7. $\frac{4a^3x}{6dy^2}$ by $\frac{2a^2x^2}{8a^2y}$.
8. $\frac{5x^2y^3z}{6a^2b^2c}$ by $\frac{2ab^2}{15xy^2}$.
9. $\frac{1}{x^2-y^2}$ by $\frac{1}{x-y}$.
10. $\frac{a-b}{x}$ by a^2-b^2 .
11. $\frac{a^2-4x^2}{a^2+4x^2}$ by $a-2x$.
12. $\frac{x^2y}{x^2-y^2}$ by $\frac{xy^2}{x+y}$.
13. $\frac{dc+dy}{2y}$ by $\frac{c+y}{xy^2}$.
14. $\frac{4}{x^2-x-2}$ by $\frac{2}{x+1}$.
15. $\frac{14}{x^3-y^3}$ by $\frac{7}{x-y}$.
16. $\frac{ax-bx}{25x^3}$ by $\frac{a-b}{5x^2}$.
17. $\frac{b^3+1}{ab}$ by $\frac{b^2-b+1}{axy}$.
18. $\frac{abx}{x^2-2x+1}$ by $\frac{bm}{x-1}$.
19. $\frac{5bc}{x^2-5x}$ by $\frac{3ac}{x^2-15x+50}$.
20. $\frac{1}{a^2-x^2}$ by $\frac{1}{(a+x)^2}$.
21. $\frac{2x}{x^2-7x}$ by $\frac{3x^2}{x^2-13x+42}$.
22. $\frac{x^2-4x+4}{x^2-2x-15}$ by $\frac{x^2-6x+8}{x^2+4x+3}$.
23. $\frac{a^2-10a+24}{a^2+5a-14}$ by $\left(\frac{a-6}{a-2} \times \frac{a-4}{a^2+6a-7}\right)$.
24. $\left(\frac{x}{2} + \frac{y}{3}\right)$ by $\left(\frac{(3x+2y)(x-y)}{6(x+y)}\right)$.
25. $4x^2-4y^2$ by $\frac{x+y}{x-y}$.

26. $\frac{x^2 - y^2}{m^3 + n^3}$ by $\frac{x - y}{m + n}$.

27. $\left(x - 4 + \frac{6}{x + 1}\right)$ by $\left(x - \frac{6}{x + 1}\right)$.

28. What is the cost of b tons of coal if 7 tons cost a dollars?

29. How many times a dollars are 2 times c dollars?

30. A man sells 4 horses at m dollars each, and after using n dollars of the money, divides the remainder equally among c children. What represents the share of each child?

31. A man exchanges d horses for f cows. At that rate what is the value of a horses?

32. A son's age is $\frac{1}{m}$ of his father's. The father's age is represented by two thirds of (a minus b) years. What represents the age of the son?

EQUATIONS.

134. Elimination by Comparison.

1. $x + 3y = 5$ and $x - y = 1$ are simultaneous equations. Transpose the term containing y in each to the second member. What are the resulting equations?

2. Since the second members of these derived equations are each equal to x , how do they compare with each other? (Ax. 1.)

3. If these second members are formed into an equation, how many unknown quantities does it contain?

4. How, then, may an unknown quantity be eliminated from two simultaneous equations by *comparison*?

1. Given $\begin{cases} 2x - y = 3 \\ x + 3y = 19 \end{cases}$ to find the values of x and y by comparison.

SOLUTION.

$$2x - y = 3 \quad (1)$$

$$x + 3y = 19 \quad (2)$$

Transposing in (1),

$$2x = 3 + y \quad (3)$$

Dividing by the coefficient of x ,

$$x = \frac{3+y}{2} \quad (4)$$

Transposing in (2),

$$x = 19 - 3y \quad (5)$$

Equating (4) and (5) (Ax. 1),

$$\frac{3+y}{2} = 19 - 3y \quad (6)$$

Clearing of fractions,

$$3 + y = 38 - 6y \quad (7)$$

Transposing, etc.,

$$7y = 35 \quad (8)$$

Dividing by the coefficient of y ,

$$y = 5 \quad (9)$$

Substituting the value of y in (5),

$$x = 19 - 15 \quad (10)$$

$$x = 4 \quad (11)$$

RULE. Find an expression for the value of the same unknown quantity in each equation. Make an equation of these values and solve it.

Solve the following by elimination by comparison :

$$2. \quad \begin{cases} 4x + y = 10 \\ x + y = 7 \end{cases}$$

$$6. \quad \begin{cases} 2x - 2y = -6 \\ 3x - 2y = 6 \end{cases}$$

$$3. \quad \begin{cases} 6x - y = 2 \\ x + 3y = 32 \end{cases}$$

$$7. \quad \begin{cases} x - y = 4 \\ 2x - y = 22 \end{cases}$$

$$4. \quad \begin{cases} x + 2y = 11 \\ 6x - y = 40 \end{cases}$$

$$8. \quad \begin{cases} 3x - y = 14 \\ 2x + 2y = 28 \end{cases}$$

$$5. \quad \begin{cases} x + 10y = 14 \\ 2x - y = 7 \end{cases}$$

$$9. \quad \begin{cases} 3x + y = 26 \\ x + 2y = 27 \end{cases}$$

$$10. \begin{cases} x + y = 9 \\ 2x + 3y = 20 \end{cases}$$

$$11. \begin{cases} 2x - y = 7 \\ 3x + 2y = 28 \end{cases}$$

$$12. \begin{cases} 3x - 2y = 2 \\ 2x - y = 5 \end{cases}$$

$$13. \begin{cases} 4x - 3y = 1 \\ 2x + y = 23 \end{cases}$$

$$14. \begin{cases} 2x + y = 27 \\ 3x - y = 3 \end{cases}$$

$$15. \begin{cases} 3x + 2y = 30 \\ 5x + y = 29 \end{cases}$$

$$16. \begin{cases} \frac{6}{x} + \frac{8}{y} = 1 \\ \frac{12}{x} + \frac{4}{y} = 1 \end{cases}$$

$$17. \begin{cases} ax + by = m \\ ax + cy = n \end{cases}$$

$$18. \begin{cases} \frac{x}{2} + \frac{y}{3} = 10 \\ \frac{x}{5} - \frac{y}{5} = -1 \end{cases}$$

$$19. \begin{cases} 2x + ay = b \\ ax + 2y = c \end{cases}$$

$$20. \begin{cases} \frac{x}{3} + \frac{y}{2} = 7 \\ \frac{x}{9} + \frac{y}{2} = 5 \end{cases}$$

$$21. \begin{cases} \frac{x}{5} + 5y = 51 \\ \frac{y}{5} + 5x = 27 \end{cases}$$

135. Elimination by Substitution.

1. In the equation $x + y = 12$, how may the value of x be found if y equals 5?

2. $x + 2y = 8$ and $2x + 3y = 13$ are simultaneous equations. Find the value of x in the first equation by transposing $2y$ to the second member.

3. If this value of x should be substituted for x in the second equation, how many unknown quantities would the resulting equation contain?

4. How, then, may an unknown quantity be eliminated from two simultaneous equations by *substitution*?

1. Given $\begin{cases} 3x - 2y = 1 \\ 4x + 3y = 24 \end{cases}$ to find the values of x and y by substitution.

$$\text{SOLUTION.} \qquad 3x - 2y = 1 \qquad (1)$$

$$4x + 3y = 24 \qquad (2)$$

$$\text{From (1),} \qquad x = \frac{1 + 2y}{3} \qquad (3)$$

$$\text{Substituting (3) in (2), } \frac{4(1 + 2y)}{3} + 3y = 24 \qquad (4)$$

$$\text{Clearing, etc.,} \qquad 4 + 8y + 9y = 72 \qquad (5)$$

$$\text{Transposing, etc.,} \qquad 17y = 68 \qquad (6)$$

$$y = 4 \qquad (7)$$

$$\text{Substituting in (3),} \qquad x = \frac{1 + 8}{3} \qquad (8)$$

$$x = 3 \qquad (9)$$

Solve the following by elimination by substitution:

$$2. \quad \begin{cases} 2x + y = 13 \\ x + 4y = 31 \end{cases}$$

$$7. \quad \begin{cases} 4x + y = 25 \\ y - 4x = 9 \end{cases}$$

$$3. \quad \begin{cases} x - 3y = 3 \\ 2x + 4y = 16 \end{cases}$$

$$8. \quad \begin{cases} 7x - 4y = 81 \\ 2x - y = 25 \end{cases}$$

$$4. \quad \begin{cases} 3x - 2y = 3 \\ 2x + 3y = 28 \end{cases}$$

$$9. \quad \begin{cases} x + 2y = 20 \\ x - 2y = 4 \end{cases}$$

$$5. \quad \begin{cases} 4x + 4y = 76 \\ 3x + y = 39 \end{cases}$$

$$10. \quad \begin{cases} 3x - y = 5 \\ 3y - x = 9 \end{cases}$$

$$6. \quad \begin{cases} 3x + y = 14 \\ x + 3y = 26 \end{cases}$$

$$11. \quad \begin{cases} 2x - y = 18 \\ x - 2y = -6 \end{cases}$$

$$12. \quad \begin{cases} 3x - 2y = 24 \\ 2x - y = 20 \end{cases}$$

$$13. \quad \begin{cases} 4x - y = 42 \\ 6x + 5y = 180 \end{cases}$$

$$14. \quad \begin{cases} 5x + 3y = 46 \\ 3x + 2y = 29 \end{cases}$$

$$15. \quad \begin{cases} 2x - y = 10 \\ 16x - 5y = 200 \end{cases}$$

$$16. \quad \begin{cases} x + \frac{y}{2} = 16 \\ \frac{x}{2} + y = 20 \end{cases}$$

$$17. \quad \begin{cases} 4x - 3y = 24 \\ 3x - 2y = 19 \end{cases}$$

$$18. \quad \begin{cases} \frac{x}{a} + \frac{y}{b} = 3 \\ \frac{x}{a} - \frac{y}{b} = 2 \end{cases}$$

$$19. \quad \begin{cases} \frac{2x}{3} - \frac{y}{2} = 0 \\ \frac{x}{7} + \frac{y}{7} = 5 \end{cases}$$

$$20. \quad \begin{cases} \frac{a}{x} + \frac{b}{y} = m \\ \frac{c}{x} + \frac{d}{y} = n \end{cases}$$

REVIEW OF EQUATIONS.

136. 1. Definition of an Equation.

2. Definition of **Members** of an Equation.

3. Definition of **Transposition**. Rule.

4. Definition of an **Axiom**. Give the Axioms.

5. Definition of **Simultaneous Equations**.

6. Definition of **Elimination**.

7. Definition of **Clearing of Fractions**. Rule.

137. When an equation contains no unknown quantity in any denominator, the highest number of factors of unknown quantities contained in any term determines the **Degree of the Equation**.

Thus, when the highest number of unknown literal factors in any term is *one*, the equation is of the *first* degree; when the highest number is *two*, the equation is of the *second* degree; when the highest number is *three*, the equation is of the *third* degree; etc.

138. An equation of the *first* degree is also called a **Simple Equation**.

Thus, $x + y = 7$, $3ax + 2a^2y = 10$ are simple equations.

139. An equation of the *second* degree is also called a **Quadratic Equation**.

Thus, $x^2 = c$, $x + xy = 4$, $abcxy = 6$ are quadratic equations.

140. Solve the following equations :

$$1. \quad x - 6 + 5x = 4 + 2x + 2.$$

$$2. \quad 4(x - 1) = 5(2x - 8).$$

$$3. \quad x(x - 4) + 6 = (x - 4)(x + 30).$$

$$4. \quad (2x + 6)(x + 1) = (2x - 2)(x + 9).$$

$$5. \quad ax - bx = a^2 - b^2.$$

$$6. \quad ax - ab = x - cx + b^2.$$

$$7. \quad a + b + 2x = c^2x.$$

$$8. \quad a^2x - a^2 - b^2 = b^2x + 2ab.$$

$$9. \quad 4x - \frac{x}{2} = \frac{x}{2} + 9.$$

$$11. \quad 4 + \frac{x}{4} = \frac{x}{2} - 3 + \frac{x}{3}.$$

$$10. \quad \frac{x}{4} - \frac{x}{5} + 3 = \frac{x}{10} + 1.$$

$$12. \quad \frac{x}{2} - \frac{x}{7} = x - 2 - \frac{x}{2}.$$

$$13. \quad \frac{x}{3} - x = \frac{x - 1}{11} - 9.$$

$$14. \quad \frac{x}{4}(2x - 6) = \frac{x}{2}(x + 4) - 7.$$

$$15. \quad \frac{11x - 80}{6} - \frac{8x - 5}{15} = 0.$$

$$16. \quad x + 3 + \frac{3(x + 3)}{7} = \frac{3(x + 3)}{2} - \frac{1}{2}.$$

$$17. \frac{2x-8}{5} - \frac{x+4}{9} = \frac{x}{2} - 5.$$

$$18. \frac{x}{a} - b = \frac{c}{d} - x.$$

$$19. \frac{ax-b}{c} + a = \frac{x+ac}{c}.$$

$$20. \frac{1}{x-1} - \frac{1}{x+1} = \frac{2x}{x^2-1}.$$

$$21. \frac{4}{x-2} + \frac{12}{x+2} = \frac{48}{x^2-4}.$$

$$22. \frac{x-12}{x-7} + \frac{x-4}{x-12} = 2 + \frac{7}{x-7}.$$

$$23. \frac{x}{3} - \frac{x^2-5x}{3x-7} = \frac{2}{3}.$$

$$24. \frac{x+7}{2} - \frac{x+5}{3} = 5 - \frac{5x+3}{4}.$$

$$25. \frac{10x+3}{3} - \frac{6x-7}{2} = 10(x-1).$$

$$26. \frac{7x+16}{21} - \frac{x}{3} = \frac{x+8}{4x-11}.$$

SUGGESTION. The equation may be expressed as follows:

$$\frac{7x}{21} + \frac{16}{21} - \frac{x}{3} = \frac{x+8}{4x-11}.$$

Simplifying,

$$\frac{16}{21} = \frac{x+8}{4x-11}.$$

$$27. \frac{20x}{25} + \frac{36}{25} + \frac{5x+20}{9x-16} = \frac{4x}{5} + \frac{86}{25}.$$

$$28. 1 - \frac{x}{x+2} = \frac{4}{x+6}.$$

$$29. \frac{5x}{x+6} + \frac{10}{x-1} = 5.$$

$$30. \frac{15}{x+3} - \frac{1}{3(x+3)} = \frac{44}{15}.$$

$$31. \frac{3x-2}{2x-5} - \frac{6x+13}{10} = \frac{21-3x}{5}.$$

$$32. \frac{7}{x-1} - \frac{6x-1}{x+1} = \frac{3(16-2x^2)}{x^2-1}$$

$$33. \frac{10x+17}{18} - \frac{12x+2}{13x-16} = \frac{5x-4}{9}.$$

See suggestion, Example 26.

$$34. \frac{18x+19}{28} + \frac{11x+21}{6x+14} = \frac{9x+15}{14}.$$

$$35. \frac{6x+1}{15} - \frac{2x-4}{7x-16} = \frac{2x-1}{5}.$$

$$36. \frac{2x-5}{16} + \frac{x+5}{4} = \frac{x+13}{8} + \frac{1}{16}.$$

$$37. \frac{4x+3}{9} = \frac{8x+19}{18} - \frac{7x-29}{5x-12}.$$

$$38. \frac{4}{x+2} + \frac{7}{x+3} = \frac{37}{x^2+5x+6}.$$

$$39. \frac{2-4x}{1-3x} + \frac{1}{6} = \frac{1-3x}{1-2x}.$$

$$40. \frac{4}{x-4} - \frac{10}{x-5} = \frac{2}{x^2-9x+20}.$$

$$41. \frac{a(x-a)}{b} + \frac{b(x-b)}{a} = x.$$

$$42. \frac{a}{bx} + \frac{b}{ax} = a^2 + b^2.$$

$$43. \frac{x^2 - a}{bx} - \frac{a - x}{b} = \frac{2x}{b} - \frac{a}{x}.$$

$$44. \frac{3ax - 2b}{3b} - \frac{ax - a}{2b} = \frac{ax}{b} - \frac{2}{3}.$$

$$45. \frac{b(c^2 + x^2)}{cx} = bd + \frac{bx}{c}.$$

$$46. \begin{cases} x + 5y = 7 \\ 4x + 3y = 11 \end{cases}$$

$$47. \begin{cases} 4x - 2y = 6 \\ 3x + 3y = 36 \end{cases}$$

$$48. \begin{cases} x - \frac{y}{2} = 8 \\ \frac{x}{2} - y = 7 \end{cases}$$

$$49. \begin{cases} \frac{x}{3} - \frac{y}{5} = 3 \\ \frac{x}{5} + \frac{y}{2} = 8 \end{cases}$$

$$50. \begin{cases} \frac{2x}{3} - \frac{y}{8} = 2 \\ \frac{x}{6} - \frac{y}{4} = -3 \end{cases}$$

$$51. \begin{cases} \frac{x}{2} + \frac{y}{2} = 2 \\ \frac{x}{4} + y = 2\frac{1}{2} \end{cases}$$

$$52. \begin{cases} x + 5y = 12 \\ \frac{x}{2} + 3y = 7 \end{cases}$$

$$53. \begin{cases} x + y = a \\ x - y = b \end{cases}$$

$$54. \begin{cases} ax + by = c \\ bx - ay = d \end{cases}$$

$$55. \begin{cases} \frac{a}{x} + \frac{b}{y} = m \\ \frac{a}{x} - \frac{b}{y} = n \end{cases}$$

$$56. \begin{cases} \frac{x}{m} + ny = m + n \\ \frac{mx}{n} + \frac{n^2y}{m} = m^2 + n^2 \end{cases}$$

$$57. \begin{cases} \frac{x}{a} + \frac{y}{b} = 1 \\ \frac{x}{b} + \frac{y}{a} = 1 \end{cases}$$

PROBLEMS.

141. 1. The sum of two numbers is 100, and the less number is 10 more than $\frac{1}{2}$ of the greater. What are the numbers?

2. A, B, and C build 216 rods of fence. A builds 7 rods a day, B builds 6 rods, and C builds $5\frac{1}{2}$ rods. If A and B work the same number of days and C works twice as many days as the others, how many days does each work?

3. A man has \$6000 in cash. He expends part of the sum for a house, and invests the remainder where it brings him an income of \$250 per year. If in 4 years the amount invested together with the income of that time equals the amount paid for the house, how much is paid for the house? How much is invested?

4. A boy bought a certain number of apples at the rate of 3 for 2 cents. He sold them at the rate of 6 for 5 cents, and gained 12 cents. How many apples did he buy?

5. A lady bought two pieces of cloth; the longer piece lacked 4 yards of being three times the length of the shorter. She paid \$2 per yard for the longer piece, and \$ $2\frac{1}{2}$ for the shorter, and the shorter piece cost just $\frac{1}{2}$ as much as the longer. How many yards were there in each piece?

6. The distance around a rectangular field is equal to 10 rods more than five times the breadth; and the length is $1\frac{3}{5}$ times the breadth. What are the length and the breadth of the field?

7. A man spends $\frac{1}{4}$ of his annual income for his board, $\frac{1}{8}$ for traveling expenses, $\frac{1}{10}$ for clothes, $\frac{1}{20}$ for other expenses, and saves \$380. What is his annual income?

8. Of a company of soldiers $\frac{3}{5}$ are on duty, $\frac{1}{2}$ of the remainder are absent on leave, $\frac{1}{10}$ of the whole are sick, and

the remaining 50 are off duty. How many soldiers are there in the company ?

9. A farm of 450 acres was divided among A, B, and C so that A had $\frac{2}{3}$ as many acres as B, and C had $\frac{1}{2}$ as many acres as A and B together. How many acres had each ?

10. A boy bought a certain number of apples at the rate of 2 for 1 cent. He sold half of the number for 1 cent apiece, and the other half at the rate of 2 for 1 cent. He gained 10 cents by the transaction. How many apples did he buy ?

SUGGESTION. Let $2x$ equal the number of apples.

11. The length of a certain field is twice the breadth. If the length were $\frac{2}{3}$ as much, and the breadth $1\frac{1}{3}$ as much, the entire distance around the field would be 64 rods. What are the length and the breadth of the field ?

12. Find a number whose half, third, and fourth added together equal the number plus 2.

13. A man in business gained \$100, and then lost $\frac{1}{2}$ of all that he had. He afterward gained \$150, when he found that his money was just equal to his original capital. What was the original capital ?

14. Two numbers are to each other as 3 to 4. If 10 is subtracted from each, the smaller one will be $\frac{2}{3}$ of the larger. What are the numbers ?

SUGGESTION. Let $3x$ and $4x$ represent the numbers.

15. A woman sold some eggs at 2 cents apiece ; but when she came to deliver them, four of the eggs being broken, she received only 192 cents. How many eggs had she ?

16. Two numbers are to each other as 5 to 7. One half of the first plus the second equals one half of the second plus 12. What are the numbers ?

17. A said to B, "Give me \$100, and I shall have as much money as you have left." B said to A, "Give me \$100, and I shall have three times as much money as you have left." How much money had each?

18. A man gained in business as follows: The first year \$400 less than his original capital; the second year \$300 less than the original capital; and the third year \$200 less than the first year. The gains of the three years amounted to \$700 more than the capital. What was the original capital?

19. A mason, 5 carpenters, and 3 assistants receive together \$191 for working a certain number of days. The mason receives \$3 per day, the carpenters \$2½, and the assistants \$1½. How many days do they work?

20. A man spent \$300 more than ½ of his earnings each year. In 5 years he had saved \$1000. How much did he earn each year?

21. A boy bought a certain number of apples at the rate of 4 for 5 cents, and sold them at the rate of 3 for 4 cents. He gained 60 cents. How many did he buy?

22. A boy spent ½ of his money and 2 cents more. He then spent ½ of what was left and 2 cents more, when he found that there remained 12 cents of his money. How much had he at first?

23. A man bought a horse and wagon, paying twice as much for the horse as for the wagon. He sold the horse at a gain of 50 per cent, and the wagon at a loss of 10 per cent. He received for both \$195. What did he pay for each?

24. A person engaged to reap a field of grain for \$1½ per acre; but leaving 6 acres not reaped, he received \$12. How many acres of grain were there?

142. Three unknown quantities.

$$1. \text{ Given } \begin{cases} 2x + y + 3z = 13 \\ 3x + 2y + z = 10 \\ x + 3y + 2z = 13 \end{cases} \text{ to find } x, y, \text{ and } z.$$

SOLUTION.

$$2x + y + 3z = 13 \quad (1)$$

$$3x + 2y + z = 10 \quad (2)$$

$$x + 3y + 2z = 13 \quad (3)$$

$$(2) \times 3, \quad 9x + 6y + 3z = 30 \quad (4)$$

$$(1), \quad 2x + y + 3z = 13$$

$$(4) - (1), \quad 7x + 5y = 17 \quad (5)$$

$$(2) \times 2, \quad 6x + 4y + 2z = 20 \quad (6)$$

$$(3), \quad x + 3y + 2z = 13$$

$$(6) - (3), \quad 5x + y = 7 \quad (7)$$

$$(7) \times 5, \quad 25x + 5y = 35 \quad (8)$$

$$(5), \quad 7x + 5y = 17$$

$$(8) - (5), \quad 18x = 18 \quad (9)$$

$$x = 1 \quad (10)$$

$$\text{Substituting (10) in (7),} \quad 5 + y = 7 \quad (11)$$

$$y = 2 \quad (12)$$

$$\text{Substituting (10) and (12) in (1), } 2 + 2 + 3z = 13 \quad (13)$$

$$z = 3 \quad (14)$$

The student will observe that by combining two of the given equations *one* of the unknown quantities is eliminated, and that by combining another pair of the given equations the same quantity is eliminated. We have thus two equations containing two unknown quantities which may be readily solved.

Find the value of each unknown quantity in the following:

$$2. \begin{cases} x + y + z = 6 \\ x + 2y + z = 7 \\ x + y + 2z = 9 \end{cases}$$

$$3. \begin{cases} 2x + 2y + z = 15 \\ 3x + y + 2z = 23 \\ x - 3y + 2z = 11 \end{cases}$$

$$4. \begin{cases} x + 2y + z = 16 \\ 4x - 3y - z = 6 \\ 2x + 2y - z = 11 \end{cases}$$

$$13. \begin{cases} x - 12y + 2z = -20 \\ 2x - y - z = 7 \\ x + z = 19 \end{cases}$$

$$5. \begin{cases} 5x - y + 2z = 38 \\ 2x + y - z = 4 \\ x - 3y + 5z = 44 \end{cases}$$

$$14. \begin{cases} 6x - y + z = 28 \\ 5x + 2y - 2z = 12 \\ 2x + 2y - z = 8 \end{cases}$$

$$6. \begin{cases} x + y + z = 15 \\ 2x + 2y - z = 6 \\ x - y + 4z = 37 \end{cases}$$

$$15. \begin{cases} \frac{x}{4} + \frac{y}{8} = 5 \\ \frac{x}{4} + \frac{z}{10} = 5 \\ \frac{y}{4} + \frac{z}{5} = 8 \end{cases}$$

$$7. \begin{cases} x + 2y - z = 6 \\ 4x - y + 2z = 27 \\ 2x + 2y + z = 31 \end{cases}$$

$$8. \begin{cases} 3x - 5y + 2z = 6 \\ 5x + y - 3z = 14 \\ 4x + 2y - 4z = -4 \end{cases}$$

$$16. \begin{cases} \frac{x}{2} + \frac{y}{5} + z = 9 \\ \frac{x}{4} + \frac{y}{2} + z = 10 \\ x + y + \frac{z}{3} = 19 \end{cases}$$

$$9. \begin{cases} x - y + z = 17 \\ x + 4y - 6z = 4 \\ 3x - 2y - 3z = 7 \end{cases}$$

$$10. \begin{cases} -2x + 2y + z = 9 \\ 5x - 3y = 7 \\ 4y + z = 31 \end{cases}$$

$$17. \begin{cases} x + y = 9 \\ x + z = 12 \\ y + z = 15 \end{cases}$$

$$11. \begin{cases} x + y = 14 \\ 2y - 3z = 9 \\ 5x - z = 5 \end{cases}$$

$$18. \begin{cases} y - x + z = -5 \\ z - y - x = -25 \\ x + y + z = 35 \end{cases}$$

$$12. \begin{cases} 2x - y = 9 \\ x - 2z = 3 \\ y - 2z = 5 \end{cases}$$

$$19. \begin{cases} 2x - 3y = 1 \\ 3y - 4z = 7 \\ 4z - 5x = -32 \end{cases}$$

$$20. \left\{ \begin{array}{l} \frac{1}{x} + \frac{1}{y} - \frac{1}{z} = \frac{2}{3} \\ \frac{1}{x} - \frac{1}{y} + \frac{1}{z} = \frac{1}{3} \\ \frac{1}{y} + \frac{1}{z} - \frac{1}{x} = 0 \end{array} \right\}$$

$$21. \left\{ \begin{array}{l} \frac{1}{x} + \frac{2}{y} = \frac{3}{5} \\ \frac{3}{y} - \frac{4}{z} = \frac{1}{5} \\ \frac{3}{z} - \frac{4}{x} = -\frac{1}{2} \end{array} \right\}$$

$$22. \left\{ \begin{array}{l} 2x - y = 6 \\ x + y - 2z = -2 \\ 4x - 3y + z = 11 \end{array} \right\}$$

$$23. \left\{ \begin{array}{l} \frac{x}{2} - \frac{z}{15} = 8 \\ \frac{z}{5} - \frac{y}{5} = 1 \\ \frac{x}{4} - \frac{y}{5} = 0 \end{array} \right\}$$

$$24. \left\{ \begin{array}{l} 25x - 20y + 15z = 80 \\ 15x - 25y + 20z = 60 \\ 20x + 15y - 25z = -40 \end{array} \right\}$$

25. A farmer has sheep in three pastures. The number in the first plus $\frac{1}{2}$ of the number in the second plus $\frac{1}{3}$ of the number in the third equals 70. The number in the first plus $\frac{1}{4}$ of the number in the second minus $\frac{1}{6}$ of the number in the third equals 30. The number in the second plus $\frac{1}{2}$ of the number in the first plus $\frac{1}{10}$ of the number in the third equals 61. How many sheep are there in each pasture?

26. Henry, James, and Ralph together have 50 cents. Henry's and Ralph's money amounts to 35 cents; James' and Ralph's amounts to 40 cents. How many cents has each?

27. A farmer has wheat, oats, and barley. The number of bushels of wheat and oats is 200; the number of bushels of wheat and barley is 190; and the number of bushels of oats and barley is 90. How many bushels are there of each kind of grain?

INVOLUTION.

- 143.** 1. What is a power of a quantity? An exponent?
2. How many times is a quantity used as a factor in producing the second power? The third power? The fourth power? The fifth power? Any power?
3. What sign has the second power of $+a$? The third power? The fourth power? Any power of a positive quantity?
4. What sign has the second power of $-a$? The third power? The fourth power? The fifth power? The sixth power? Any even power of a negative quantity? Any odd power of a negative quantity?
5. Which powers of a negative quantity are positive? Which are negative?
6. What is the third power of a^2 ? The fourth power? The fifth power? The sixth power? The n th power?
7. What is the third power of a^3 ? The fourth power? The fifth power? The sixth power? The n th power?
8. How is the exponent of a power of a quantity determined from the exponent of the quantity?

144. The process of finding a power of a quantity is called **Involution**.

145. PRINCIPLES. 1. *All powers of a positive quantity are positive.*

2. *All even powers of a negative quantity are positive, and all odd powers are negative.*

3. *The exponent of the power of a quantity is equal to the exponent of the quantity multiplied by the exponent of the power to which the quantity is to be raised.*

146. Involution of monomials.

1. What is the fourth power of $3a^2x^3$?

SOLUTION. $(3a^2x^3)^4 = 3a^2x^3 \cdot 3a^2x^3 \cdot 3a^2x^3 \cdot 3a^2x^3 = 81a^8x^{12}$.

RULE. *Raise the numerical coefficient to the required power; multiply the exponent of each literal quantity by the exponent of the power to which it is to be raised, and prefix the proper sign to the result.*

2. What is the third power of $\frac{2ab^2}{3x^2y^3}$?

SOLUTION. $\left(\frac{2ab^2}{3x^2y^3}\right)^3 = \frac{2ab^2}{3x^2y^3} \times \frac{2ab^2}{3x^2y^3} \times \frac{2ab^2}{3x^2y^3} = \frac{8a^3b^6}{27x^6y^9}$.

In raising a fraction to a power, both numerator and denominator must be raised to the required power.

Find the values of the following:

- | | | |
|---------------------|----------------------|------------------------|
| 3. $(2x^2y)^2$. | 10. $(5x^4y^2)^3$. | 17. $(-2a^3b^2)^7$. |
| 4. $(4xyz^2)^2$. | 11. $(3a^2bc^2)^4$. | 18. $(-10ax^3b)^5$. |
| 5. $(-3x^3z)^2$. | 12. $(-2m^2n)^4$. | 19. $(a^2b^2c^5d)^9$. |
| 6. $(2a^2b^2)^3$. | 13. $(-4am^3)^5$. | 20. $(-a^3b^4m)^8$. |
| 7. $(-4ab^2x)^3$. | 14. $(2ab^2d)^5$. | 21. $(-2a^5b^2de)^7$. |
| 8. $(-3c^2d^2)^2$. | 15. $(-6a^2m)^3$. | 22. $(-x^4y^3z^7)^9$. |
| 9. $(-x^3yz)^3$. | 16. $(2x^4y)^8$. | 23. $(x^2y^mz^n)^5$. |

- | | | |
|---|---|---|
| 24. $(2 x^4 y^4 z^8)^6.$ | 32. $\left(-\frac{x^2 y z^3}{2 a m^3 n}\right)^4.$ | 37. $\left(\frac{a^3 x^n}{y^4 z^5}\right)^8.$ |
| 25. $(-2 b^2 c^m d^{2n})^7.$ | 33. $\left(\frac{a^2 b^2 c^n}{x^n y^2 z}\right)^5.$ | 38. $\left(\frac{8 a^2 b}{7 x^2 y}\right)^4.$ |
| 26. $(a^3 b^{2x} c^{3x})^8.$ | 34. $\left(\frac{5 a^2 x y}{6 m n^2 z}\right)^3.$ | 39. $\left(\frac{a^3 x^4}{b^4 y^3}\right)^{3n}.$ |
| 27. $(5 a^2 b y^3 z^4)^4.$ | 35. $\left(\frac{a^2 b^2 c^3}{x^4 y^4 z^2}\right)^n.$ | 40. $\left(-\frac{x^4 y}{10 a^2 m^3}\right)^7.$ |
| 28. $(-a^4 y^{5n} z^{3n})^3.$ | 36. $\left(\frac{3 a^3 x^4 y}{4 m^2 n z^4}\right)^6.$ | 41. $\left(\frac{b^2 c d^4}{m^2 n^3 x^5}\right)^9.$ |
| 29. $(2 a^2 b x y)^n.$ | | |
| 30. $\left(\frac{2 a b}{3 m x}\right)^2.$ | | |
| 31. $\left(\frac{3 x^2 y}{4 a^2 z^2}\right)^3.$ | | |

147. Involution of polynomials.

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2.$$

$$(a - b)^2 = (a - b)(a - b) = a^2 - 2ab + b^2.$$

$$(a + b)^3 = (a + b)(a + b)(a + b) = a^3 + 3a^2b + 3ab^2 + b^3.$$

$$(a - b)^3 = (a - b)(a - b)(a - b) = a^3 - 3a^2b + 3ab^2 - b^3.$$

$$(a - b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$$

$$(a - b)^5 = a^5 - 5a^4b + 10a^3b^2 - 10a^2b^3 + 5ab^4 - b^5.$$

$$\begin{aligned}(a - b + c)^2 &= [(a - b) + c]^2 = (a - b)^2 + 2(a - b)c + c^2 \\ &= a^2 - 2ab + b^2 + 2ac - 2bc + c^2.\end{aligned}$$

Raise to the required power:

- | | | |
|-------------------|--------------------|-------------------------------|
| 1. $(x - y)^2.$ | 8. $(m - 3n)^2.$ | 15. $(a^2 + b^2)^2.$ |
| 2. $(3a + 2b)^2.$ | 9. $(x + 2y)^2.$ | 16. $(a + b + c)^2.$ |
| 3. $(5a + 4c)^2.$ | 10. $(2a - 5b)^2.$ | 17. $(a + b - c)^2.$ |
| 4. $(x + y)^3.$ | 11. $(3x + 2y)^3.$ | 18. $(x - y - z)^2.$ |
| 5. $(x - y)^3.$ | 12. $(4x - 3y)^3.$ | 19. $(2x + 2y - z)^2.$ |
| 6. $(a + b)^4.$ | 13. $(a + b)^5.$ | 20. $(a^n + b^n + c^n)^2.$ |
| 7. $(x - y)^4.$ | 14. $(x + 1)^3.$ | 21. $(a^m - b^m - c^{2m})^3.$ |

EVOLUTION.

148. 1. What are the two equal factors of 25? Of 36? Of 64?

2. What is one of the two equal factors, or what is the second or square root of 25? Of 81? Of 100?

3. What is one of the three equal factors, or what is the third or cube root of a^3 ? Of $8a^3$? Of $27a^3$? Of $8a^6$?

4. What is the sign of any power of a positive quantity? Since any power of a positive quantity is positive, what sign may one of the equal factors, or a root of a positive quantity have?

5. What is the sign of the second power of a negative quantity? The fourth power? The sixth power? Any even power?

6. Since any even power of a positive or negative quantity has the positive sign, what is the sign of the *even* root of any quantity?

7. What sign has the second power of -3 ? The third power? The fourth power? The fifth power? What powers of a negative quantity are negative?

8. Since a power having the negative sign is the product of an odd number of equal negative factors, what is the sign of the odd root of a negative quantity?

9. What kind of quantity used twice as a factor will give a product with the negative sign? What kind, used four times? Any even number of times?

10. Since no quantity used an even number of times as a factor will give a negative product, what may be said regarding an even root of a negative quantity?

149. One of the equal factors of a quantity is called the **Root** of the quantity.

Thus, 3 is a root of 9, of 27, etc.; a is a root of a^3 , of a^4 , etc.

150. Roots are named from the number of equal factors into which the quantity is resolved or separated.

Thus, one of the *two* equal factors is called the *second* or *square* root; one of the *three* equal factors, the *third* or *cube* root; one of the *four* equal factors, the *fourth* root, etc.

151. The root of a quantity is indicated by placing the sign $\sqrt{}$, called the **Radical Sign**, before the quantity.

A number or quantity called the **Index** is frequently written at the opening of the radical sign, to show what root is sought.

Thus, $\sqrt[3]{a}$ shows that the *third* root of a is sought; $\sqrt[5]{a}$, the *fifth* root, etc.

When no index is written in the opening of the radical sign, the second or square root is indicated.

152. The process of finding the root of a quantity is called **Evolution**.

153. PRINCIPLES. 1. *An odd root of a quantity has the same sign as the quantity itself.*

2. *An even root of a positive quantity is either positive or negative.*

3. *An even root of a negative quantity is impossible or imaginary.*

154. Evolution of a monomial.

1. What is the square root of $16 a^4 x^6$?

PROCESS.

$$\sqrt{16 a^4 x^6} = \pm 4 a^2 x^3$$

EXPLANATION. Since in squaring a monomial we square the coefficient and multiply the exponents of the letters by 2, to extract the square root we must

extract the square root of the coefficient, and divide the exponents of the letters by 2.

The sign of the root is either plus or minus (Prin. 2). Hence the square root of the quantity is $\pm 4 a^2 x^3$.

RULE. *Extract the required root of the numerical coefficient; divide the exponent of each letter by the index of the root sought; and prefix the proper sign to the result.*

The root of a fraction is found by taking the root of the numerator and of the denominator separately.

Find the values of the following:

- | | |
|--|---|
| 2. $\sqrt{4 a^2 b^2}$. | 13. $\sqrt[7]{x^{14} y^{35} z^{28}}$. |
| 3. $\sqrt[3]{8 x^3 y^6}$. | 14. $\sqrt[3]{-64 x^{3n} y^9 z^{12n}}$. |
| 4. $\sqrt{25 x^4 y^4}$. | 15. $\sqrt{100 a^4 b^2 c^{10}}$. |
| 5. $\sqrt[3]{-27 m^3 n^6}$. | 16. $\sqrt[3]{-343 m^6 x^9 z^{12}}$. |
| 6. $\sqrt[4]{16 a^4 b^8 c^{20}}$. | 17. $\sqrt[4]{x^8 y^{20} z^{12}}$. |
| 7. $\sqrt[5]{-a^{10} b^{30} c^{15}}$. | 18. $\sqrt[6]{a^{6n} b^{24} c^{12n}}$. |
| 8. $\sqrt[3]{-125 x^9 y^3 z^{15}}$. | 19. $\sqrt[5]{-32 m^{10} n^{15} x^{25} y^{40}}$. |
| 9. $\sqrt[6]{a^{12} b^{24} c^6 d^{18}}$. | 20. $\sqrt{\frac{4 a^2 b^4}{9 x^4 y^6}}$. |
| 10. $\sqrt{49 m^8 n^{10}}$. | 21. $\sqrt{\frac{25 m^4 n^8}{36 a^{10} b^6}}$. |
| 11. $\sqrt[4]{81 a^{12} b^{16} e^4}$. | 22. $\sqrt[3]{\frac{8 a^3 b^9 c^{15}}{27 x^6 y^{12} z^{21}}}$. |
| 12. $\sqrt[5]{-32 a^{10} x^{15} y^{20}}$. | |

$$23. \sqrt[3]{-\frac{27 a^6 b^9}{343 x^{15} y^{24}}}$$

$$25. \sqrt{\frac{81 x^{10} y^{12}}{144 m^{16} n^{20}}}$$

$$24. \sqrt[5]{\frac{x^{10} y^{15} z^{25}}{a^{20} b^{30} c^{35}}}$$

$$26. \sqrt[4]{\frac{16 a^8 b^{12} c^{32}}{256 x^{16} y^{24} z^8}}$$

155. To extract the square root of a polynomial.

1. Since $a^2 + 2ab + b^2$ is the square of $(a + b)$, what is the square root of $a^2 + 2ab + b^2$?

2. How may the first term of the square root be found from the first term of $a^2 + 2ab + b^2$?

3. Since the first term of the root is found to be a , how may the second term be found from $2ab$, the second term of the power?

1. Find the square root of $a^2 + 2ab + b^2$.

PROCESS.

$$\begin{array}{rcl}
 & & a^2 + 2ab + b^2 \quad \underline{a + b} \\
 & & \underline{a^2} \\
 \text{Trial divisor} & = & 2a \quad \underline{2ab + b^2} \\
 \text{Complete divisor} & = & 2a + b \quad \underline{2ab + b^2}
 \end{array}$$

EXPLANATION. The quantities are arranged according to the descending powers of a . Since the first term of the quantity is a^2 , its square root a is the first term of the root sought. Subtracting a^2 from the quantity, there is a remainder of $2ab + b^2$.

Since the first term of the remainder is $2ab$, if it is divided by $2a$, twice the root already found, the quotient b will be the second term of the root. $2a$, or twice the root found, is called the *trial divisor*. The *complete divisor*, or the divisor which multiplied by b will produce the remainder $2ab + b^2$, is, therefore, $2a + b$. It is formed by adding to the trial divisor the second term of the root. Multiplying the complete divisor by b and subtracting, there is no remainder. Therefore $a + b$ is the square root of the quantity.

2. Find the square root of $x^4 - 6x^3 + 13x^2 - 12x + 4$.

PROCESS.

$$\begin{array}{r}
 x^4 - 6x^3 + 13x^2 - 12x + 4 \quad | \quad x^2 - 3x + 2 \\
 \underline{x^4} \\
 2x^2 - 3x \quad | \quad - 6x^3 + 13x^2 \\
 \quad | \quad - 6x^3 + 9x^2 \\
 \hline
 2x^2 - 6x + 2 \quad | \quad 4x^2 - 12x + 4 \\
 \quad | \quad 4x^2 - 12x + 4 \\
 \hline
 \quad | \quad 0
 \end{array}$$

EXPLANATION. The process is the same as in the previous example until the first two terms of the root are found.

To find the third term of the root we consider $x^2 - 3x$ as *one quantity*, which we multiply by 2 for the *trial divisor*. Dividing the first term of the remainder by the first term of this trial divisor, the third term of the root is obtained, which is $+2$. Annexing $+2$ to the trial divisor already found, the *complete divisor* $2x^2 - 6x + 2$ is obtained. This when multiplied by 2 gives as a product $4x^2 - 12x + 4$, which subtracted from $4x^2 - 12x + 4$ leaves no remainder. Therefore, the square root is $x^2 - 3x + 2$.

RULE. Arrange the terms of the polynomial with reference to the consecutive powers of some letter.

Extract the square root of the first term, write the result as the first term of the root, and subtract its square from the given polynomial.

Divide the first term of the remainder by twice the root already found, as a trial divisor, and the quotient will be the next term of the root. Write this result in the root, and annex it to the trial divisor to form a complete divisor.

Multiply the complete divisor by this term of the root, and subtract the product from the first remainder.

Find the next term of the root by dividing the first term of the remainder by the first term of the trial divisor.

Form the complete divisor as before, and continue in this manner until all the terms of the root are found.

Find the square root of the following :

3. $x^2 + 4xy + 4y^2$.
4. $a^2 - 6a + 9$.
5. $16x^2 + 24xy + 9y^2$.
6. $9x^2 - 12xy + 4y^2$.
7. $25m^2 + 60m + 36$.
8. $100a^2 - 20a + 1$.
9. $81a^2 + 90a + 25$.
10. $36x^2 + 96xy + 64y^2$.
11. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc$.
12. $x^2 + 2xy + 8x + y^2 + 8y + 16$.
13. $x^2 + 4y^2 - 4xy - 4xz + 4z^2 + 8yz$.
14. $4a^2 + 9b^2 + 4 - 12ab + 8a - 12b$.
15. $4a^2 + 4ab - 4a + b^2 - 2b + 1$.
16. $a^4 - 2a^3 - 3a^2 + 4a + 4$.
17. $4x^4 + 4x^3 + 5x^2 + 2x + 1$.
18. $9a^2 + c^2 - 6ac + 30a - 10c + 25$.
19. $16m^4 + 32m^3 - 16m^2 - 32m + 16$.
20. $30a^3 + 39a^2 + 25a^4 + 18a + 9$.
21. $11m^2 + 6m^3 + m^4 + 6m + 1$.
22. $4 + 4x - 4y + x^2 + y^2 - 2xy$.
23. $m^4 + 6m^3 + 9m^2 - 4m^2n - 12mn + 4n^2$.
24. $4a^4 - 4a^2b + b^2 + 24a^2 - 12b + 36$.
25. $x^4 - 4x^3 + 5x^2 - 2x + \frac{1}{4}$.
26. $x^2 + 2xy - 2xz - 2yz + y^2 + z^2$.

156. To extract the square root of numbers.

$$\begin{array}{lll}
 1^2 = 1 & 10^2 = 100 & 100^2 = 10000 \\
 9^2 = 81 & 99^2 = 9801 & 999^2 = 998001
 \end{array}$$

The student should observe carefully the number of figures required to express the square of units, tens, and hundreds in the above examples.

The square of the smallest and of the largest number of each order is given, consequently the number of figures required to express the square of a number can be readily discovered.

1. How many figures are required to express the square of any number of *units* ?

2. How does the number of figures required to express the second power of any number between 9 and 100 compare with the number of figures in the number ?

3. How does the number of figures expressing the second power of any number between 99 and 1000 compare with the number of figures in the number ?

4. If the second power of a number is expressed by 3 figures, how many orders of units are there in the number ?

If by 4, how many ? By 5 ? By 7 ?

157. PRINCIPLES. 1. *The square of a number is expressed by twice as many figures as is the number itself, or by one less than twice as many.*

2. *The orders of units in the square root of a number correspond to the number of periods of two figures each into which the number can be separated, beginning at units.*

The left-hand period may contain only one figure.

158. If the tens of a number are represented by t , and the units by u , the square of a number consisting of *tens* and *units* will be the square of $(t + u)$ or $t^2 + 2tu + u^2$.

Thus, $35 = 3$ tens plus 5 units, or $30 + 5$, and $35^2 = 30^2 + 2(30 \times 5) + 5^2 = 1225$.

1. What is the square root of 1296?

PROCESS.

$$\begin{array}{r}
 12 \cdot 96 \overline{) 30 + 6} \\
 t^2 = \quad 900 \\
 2t = 60 \overline{) 396} \\
 u = 6 \\
 \hline
 2t + u = 66 \overline{) 396}
 \end{array}$$

EXPLANATION. According to Prin.

2, the orders of units in the square root of a number may be determined by separating the number into periods of two figures each, beginning at units. Separating 1296 thus, there are found to be two orders of units in the root; that is, it is composed of tens and units. Since the square of tens is

hundreds, and the hundreds of the power are less than 16 or 4^2 , and more than 9 or 3^2 , the tens' figure of the root must be 3. 3 tens, or 30, squared is 900, and 900 subtracted from 1296 leaves 396, which is equal to 2 times the tens multiplied by the units plus the units squared. Since 2 times the tens multiplied by the units is much greater than the square of the units, 396 is *nearly* 2 times the tens multiplied by the units. Therefore, if 396 is divided by 2 times the tens, or 60, the quotient will be approximately the units of the root. Dividing by 60 the *trial divisor*, the units are found to be 6. And since the *complete divisor* is found by adding the units to twice the tens, the complete divisor is $60 + 6$, or 66. This multiplied by 6 gives as a product 396, which, when subtracted from 396, leaves no remainder. Therefore, the square root of 1296 is 36.

Any number may be regarded as composed of tens and units; hence the process given above has a general application.

Thus, 495 may be considered as 49 tens and 5 units; 3843 as 384 tens and 3 units.

2. Find the square root of 61009.

SOLUTION.

$$\begin{array}{r}
 6 \cdot 10 \cdot 09 \overline{) 247} \\
 4 \\
 \hline
 \text{Trial divisor} = 2 \times 20 = 40 \quad 210 \\
 \text{Complete divisor} = 40 + 4 = 44 \quad 176 \\
 \hline
 \text{Trial divisor} = 2 \times 240 = 480 \quad 3409 \\
 \text{Complete divisor} = 480 + 7 = 487 \quad 3409
 \end{array}$$

RULE. *Separate the number into periods of two figures each, beginning at units.*

Find the greatest square in the left-hand period, and write its root for the first figure of the required root.

Square this root, subtract the result from the left-hand period, and annex to the remainder the next period for a new dividend.

Double the root already found, with a cipher annexed for a trial divisor, and by it divide the dividend. The quotient, or quotient diminished, will be the second figure of the root. Add to the trial divisor the figure last found, multiply this complete divisor by the figure of the root found, subtract the product from the dividend, and to the remainder annex the next period for the next dividend.

Proceed in this manner until all the periods have been used. The result will be the square root sought.

Decimals are pointed off into periods of two figures each, by beginning with tenths and passing to the right.

Extract the square root of the following:

3. 729.	12. 9409.	21. 978121.
4. 6561.	13. 55225.	22. .004096.
5. 9604.	14. 26569.	23. 19.1844.
6. 5329.	15. 13924.	24. 620.01.
7. 7225.	16. 103041.	25. .00632025.
8. 4356.	17. 180625.	26. 12.6025.
9. 2916.	18. 253009.	27. 17.3056.
10. 2401.	19. 185761.	28. 331.9684.
11. 9025.	20. 265225.	29. 13.7641.

159. To extract the cube root of a polynomial.

1. Since $a^3 + 3a^2b + 3ab^2 + b^3$ is the third power or cube of $(a + b)$, what is the third or cube root of $a^3 + 3a^2b + 3ab^2 + b^3$?

2. How may the first term of the root be found from the first term of the power, a^3 ?

3. Since the first term of the root is known to be a , how may the second term of the root be found from the second term of the power, $3a^2b$?

4. Since $a^3 + 3a^2b + 3ab^2 + b^3$ is the cube of $(a + b)$, after the cube of the first term of the root has been subtracted from $a^3 + 3a^2b + 3ab^2 + b^3$, by what quantities must b , the second term, be multiplied to make the entire power?

1. Find the cube root of $x^3 + 3x^2y + 3xy^2 + y^3$.

PROCESS.

$$\begin{array}{r}
 x^3 + 3x^2y + 3xy^2 + y^3 \overline{) x + y} \\
 x^3 \\
 \hline
 \text{Trial divisor} \quad = 3x^2 \overline{) 3x^2y + 3xy^2 + y^3} \\
 \text{Complete divisor} = 3x^2 + 3xy + y^2 \overline{) 3x^2y + 3xy^2 + y^3}
 \end{array}$$

EXPLANATION. Since, if the quantity is a cube, one of the terms is a cube, the first term of the root is the cube root of x^3 , which is x . Subtracting x^3 from the quantity, there is left $3x^2y + 3xy^2 + y^3$. Since the second term of the root can be found from the first term of the remainder by dividing it by three times the square of the root already found, the second term of the root of the quantity will be found by dividing $3x^2y$ by $3x^2$, the *trial divisor*, which gives y for the second term of the root. Since the last three terms of the cube of a binominal are composed of the product of the second term by three times the square of the first, three times the product of the first multiplied by the square of the second and the cube of the second, $3x^2 + 3xy + y^2$ is the *complete divisor*.

The complete divisor multiplied by y is $3x^2y + 3xy^2 + y^3$, and this subtracted from the given quantity leaves no remainder. Therefore, $x + y$ is the cube root of $x^3 + 3x^2y + 3xy^2 + y^3$.

2. Find the cube root of $a^6 - 3a^5 + 6a^4 - 7a^3 + 6a^2 - 3a + 1$.

SOLUTION.

$$\begin{array}{r}
 a^6 - 3a^5 + 6a^4 - 7a^3 + 6a^2 - 3a + 1 \overline{) a^2 - a + 1} \\
 \underline{a^6} \\
 -3a^5 + 6a^4 - 7a^3 \\
 \underline{-3a^5 + 3a^4 - a^3} \\
 3a^4 - 6a^3 + 3a^2 \\
 \underline{3a^4 - 6a^3 + 6a^2 - 3a + 1} \\
 3a^4 - 6a^3 + 6a^2 - 3a + 1 \overline{) 3a^4 - 6a^3 + 6a^2 - 3a + 1}
 \end{array}$$

EXPLANATION. The first two terms of the root are found as in Ex. 1.

Since $(a^2 - a + 1)^3 = \{(a^2 - a) + 1\}^3$, $a^2 - a + 1$ may be conveniently cubed as a binomial whose first term is $(a^2 - a)$. Hence, in extracting the cube root of the cube of $a^2 - a + 1$, the first two terms of the root when found, are grouped into one term, $(a^2 - a)$, and regarded as the *first term of the root*. The next term of the root may then be regarded as the second term and found as in Ex. 1.

RULE. *Arrange the polynomial with reference to the consecutive powers of some letter.*

Extract the cube root of the first term, write the result as the first term of the root, and subtract its cube from the given polynomial.

Divide the first term of the remainder by three times the square of the root already found, as a trial divisor, and the quotient will be the next term of the root.

Add to this trial divisor three times the product of the first and second terms of the root, and the square of the second term. The result will be the complete divisor.

Multiply the complete divisor by the last term of the root found, and subtract this product from the dividend.

Find the next term of the root by dividing the first term of the remainder by the first term of the trial divisor.

Form the complete divisor as before, and continue in this manner until all the terms of the root are found.

Find the cube root of the following :

3. $a^3 + 3a^2b + 3ab^2 + b^3$.

5. $x^3 + 6x^2 + 12x + 8$.

4. $x^3 - 3x^2y + 3xy^2 - y^3$.

6. $27a^3 + 27a^2 + 9a + 1$.

7. $27 x^3 - 27 x^2 y + 9 x y^2 - y^3$.
8. $x^6 + 12 x^4 + 48 x^2 + 64$.
9. $8 m^3 - 12 m^2 n + 6 m n^2 - n^3$.
10. $x^6 + 3 x^5 + 3 x^4 + x^3$.
11. $a^3 m^3 + 6 a^2 m^2 b + 12 a m b^2 + 8 b^3$.
12. $8 a^6 - 60 a^4 c^2 + 150 a^2 c^4 - 125 c^6$.
13. $27 + 54 b + 36 b^2 + 8 b^3$.
14. $1 + 3 a + 3 a^2 + a^3$.
15. $m^6 + 18 m^4 + 108 m^2 + 216$.
16. $8 x^3 + 36 x^2 y + 54 x y^2 + 27 y^3$.
17. $64 x^9 + 576 x^6 y^2 + 1728 x^3 y^4 + 1728 y^6$.
18. $x^6 - 3 a x^5 + 5 a^3 x^3 - 3 a^5 x - a^6$.
19. $8 x^3 - 84 x^2 + 294 x - 343$.
20. $343 x^3 + 588 x^2 y + 336 x y^2 + 64 y^3$.
21. $a^9 + 9 a^6 b^3 + 27 a^3 b^6 + 27 b^9$.
22. $8 x^6 + 72 x^4 y^2 + 216 x^2 y^4 + 216 y^6$.
23. $a^6 b^3 - 9 a^4 b^2 + 27 a^2 b - 27$.
24. $1 + 9 x + 27 x^2 + 27 x^3$.
25. $m^{12} - 3 m^8 n + 3 m^4 n^2 - n^3$.
26. $x^6 - 6 x^5 + 15 x^4 - 20 x^3 + 15 x^2 - 6 x + 1$.
27. $a^6 - 9 a^5 + 33 a^4 - 63 a^3 + 66 a^2 - 36 a + 8$.
28. $x^6 - 3 x^5 + 5 x^3 - 3 x - 1$.
29. $27 m^6 - 135 m^5 + 171 m^4 + 55 m^3 - 114 m^2 - 60 m - 8$.
30. $8 x^6 + 36 x^5 y + 42 x^4 y^2 - 9 x^3 y^3 - 21 x^2 y^4 + 9 x y^5 - y^6$.
31. $1 - 9 x + 39 x^2 - 99 x^3 + 156 x^4 - 144 x^5 + 64 x^6$.
32. $1 + 12 x + 60 x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64 x^6$.

160. To extract the cube root of numbers.

$1^3 = 1$	$10^3 = 1000$	$100^3 = 1000000$
$3^3 = 27$	$36^3 = 46656$	$361^3 = 47044881$
$9^3 = 729$	$99^3 = 970299$	$999^3 = 997002999$

The student should observe carefully the number of figures required to express the cube of units, tens, and hundreds in the above examples.

The cube of the smallest, of the largest, and of an intermediate number of each order is given, consequently the number of figures required to express the cube of any number may be readily discovered.

1. How many figures are required to express the cube of any number of *units*?

2. How does the number of figures required to express the cube of any number between 9 and 100 compare with the number of figures expressing the number?

3. How does the number of figures required to express the cube of any number between 99 and 1000 compare with the number of figures expressing the number?

4. If, then, the cube of a number is expressed by 4 figures, how many orders of units are there in the root? If by 5 figures, how many? If by 6 figures, how many? If by 8 figures, how many?

5. How may the number of figures in the cube root of a number be found?

161. PRINCIPLES. 1. *The cube of a number is expressed by three times as many figures as the number itself, or by one or two less than three times as many.*

2. *The orders of units in the cube root of a number correspond to the number of periods of three figures each into which the number can be separated, beginning at units.*

The left-hand period may contain one, two, or three figures.

162. If the tens of a number are represented by t , and the units by u , the cube of a number consisting of tens and units will be the cube of $(t + u)$ or $t^3 + 3t^2u + 3tu^2 + u^3$.

Thus, $35 = 3 \text{ tens} + 5 \text{ units}$, or $30 + 5$, and $35^3 = 30^3 + 3(30^2 \times 5) + 3(30 \times 5^2) + 5^3 = 42875$.

1. What is the cube root of 74088 ?

PROCESS.

		74·088	40 + 2
	$t^3 =$	64 000	
Trial divisor	$3t^2 = 4800$	10088	
	$3tu = 240$		
	$u^2 = 4$		
Complete divisor	$= 5044$	10088	

EXPLANATION. According to Prin. 2, the orders of units may be determined by separating the number into periods of three figures each, beginning at units. Separating 74088 thus, there are found to be two orders of units in the root; that is, it is composed of tens and units. Since the cube of tens is thousands, and the thousands of the power are less than 125 or 5^3 , and more than 64 or 4^3 , the tens' figure of the root must be 4. 4 tens, or 40, cubed is 64000, and 64000 subtracted from 74088 leaves 10088, which is equal to three times the $\text{tens}^2 \times \text{the units} + 3 \text{ times the tens} \times \text{the units}^2 + \text{the units}^3$.

Since three times the tens^2 is much greater than three times the tens multiplied by the square of the units, 10088 is a little more than three times the tens squared multiplied by the units. If, then, 10088 is divided by 3 times the square of the tens, or 4800, the *trial divisor*, the quotient 2 will be the units of the root, provided proper allowance has been made for the additions necessary to obtain the complete divisor. 4800 is contained in 10088 twice; consequently the second figure of the root or the units is 2.

Since the complete divisor is found by adding to three times the square of the tens, three times the tens multiplied by the units, and the square of the units, the *complete divisor* will be $4800 + 240 + 4$ or 5044. This multiplied by 2 gives as a product 10088, which, subtracted from 10088, leaves no remainder. Therefore, the cube root of 74088 is 42.

Since any number may be regarded as composed of tens and units, the process given above has a general application.

2. What is the cube root of 1860867 ?

SOLUTION.

	1 · 860 · 867 <u>123</u>
	1
Trial divisor = $3(10)^2 = 300$	860
$3(10 \times 2) = 60$	
$2^2 = 4$	
Complete divisor = 364	728
Trial divisor = $3(120)^2 = 43200$	132867
$3(120 \times 3) = 1080$	
$3^2 = 9$	
Complete divisor = 44289	132867

RULE. Separate the number into periods of three figures each, beginning at units.

Find the greatest cube in the left-hand period, and write its root for the first figure of the required root. Cube the root, subtract the result from the left-hand period, and annex to the remainder the next period for a dividend.

Take three times the square of the root already found with a cipher annexed, for a trial divisor, and by it divide the dividend. The quotient or quotient diminished will be the second figure of the root. To this trial divisor add three times the product of the first part of the root with a cipher annexed, multiplied by the second part, and also the square of the second part. Their sum will be the complete divisor.

Multiply the complete divisor by the second part of the root, and subtract the product from the dividend. Continue thus until all the figures of the root have been found.

Decimals are pointed off into periods of three figures each, by beginning at tenths and passing to the right.

3. What is the cube root of 95256152263?

SOLUTION.

$$\begin{array}{r}
 95 \cdot 256 \cdot 152 \cdot 263 \overline{)4567} \\
 64 \\
 \hline
 4800 \overline{)31256} \\
 600 \\
 25 \\
 \hline
 5425 \overline{)27125} \\
 4131152 \\
 \hline
 607500 \\
 8100 \\
 36 \\
 \hline
 615636 \overline{)3693816} \\
 437336263 \\
 \hline
 62380800 \\
 95760 \\
 49 \\
 \hline
 62476609 \overline{)437336263}
 \end{array}$$

An abridged method of extracting the cube root of a number is presented on page 339 of the author's Standard Arithmetic.

Extract the cube root of the following :

- | | | |
|-------------|----------------|-----------------|
| 4. 571787. | 12. 16581375. | 20. .015625. |
| 5. 148877. | 13. 44361864. | 21. 43614208. |
| 6. 262144. | 14. 100544625. | 22. 13312053. |
| 7. 250047. | 15. 5545233. | 23. .004019679. |
| 8. 704969. | 16. 34965783. | 24. .000166375. |
| 9. 912673. | 17. 41063625. | 25. 28.094464. |
| 10. 314432. | 18. 68417929. | 26. 130323.843. |
| 11. 614125. | 19. 743677416. | 27. 48.228544. |

QUADRATIC EQUATIONS.



PURE QUADRATIC EQUATIONS.

163. An equation of the second degree is called a **Quadratic Equation**.

164. A quadratic equation which contains only the second power of the unknown quantity is called a **Pure Quadratic Equation**.

165. A value of the unknown quantity in an equation is called a **Root** of the equation.

1. Given $2x^2 - 5 = 45$, to find the value of x .

SOLUTION.

$$2x^2 - 5 = 45$$

$$2x^2 = 45 + 5$$

$$2x^2 = 50$$

$$x^2 = 25$$

Extracting the square root, $x = \pm 5$

2. Given $ax^2 + c = bx^2 + d$, to find the value of x .

SOLUTION.

$$ax^2 + c = bx^2 + d$$

$$ax^2 - bx^2 = d - c$$

$$(a - b)x^2 = d - c$$

$$x^2 = \frac{d - c}{a - b}$$

Extracting the square root, $x = \pm \sqrt{\frac{d - c}{a - b}}$

Find the values of x in the following equations:

3. $x^2 - 4 = 12$.

12. $15x^2 - 6 = 7x^2 + 194$.

4. $x^2 + 1 = 10$.

13. $x(x-4) = -x^2 - 4x + 8$.

5. $2x^2 - 5 = x^2 + 20$.

14. $\frac{x-2}{5} = \frac{12}{x+2}$.

6. $3x^2 - 15 = 57 + x^2$.

7. $x^2 + 1 = \frac{x^2}{4} + 4$.

15. $2x^2 - 8 = \frac{x^2}{3} + 7$.

8. $5x^2 - 2 = 2x^2 + 25$.

16. $x(2x+3) = x^2 + 3x + 81$.

9. $x^2 + 18 = 22 - 3x^2$.

17. $\frac{x^2+2}{3} + 8 = 42$.

10. $3x^2 - 29 = \frac{x^2}{4} + 510$.

18. $x^2 - 9 = a^2 - 6a$.

11. $8x^2 + 2 = 42 - 2x^2$.

19. $\frac{x^2}{a} = 4a$.

20. $x(x-2a) = 2x^2 - 2ax - a^2$.

PROBLEMS.

166. 1. If to four times the square of a certain number 3 is added, the sum is 19. What is the number?

2. Two numbers are to each other as 2 to 5, and the sum of their squares is 261. What are the numbers?

SUGGESTION. Let $2x$ and $5x$ represent the numbers.

3. If a certain number is increased by 1 and also diminished by 1, the product of the sum and difference is 48. What is the number?

4. The area of one square field is twice that of another, and both together contain 867 square rods. What is the length of a side of the smaller?

SUGGESTION. Let x represent length and x^2 area of smaller field.

5. A lady paid \$36 for a certain number of yards of cloth. She paid $\frac{1}{4}$ as many dollars per yard as there were yards in the piece. How many yards were there?

6. What number is that whose square plus 18 is equal to half its square plus $30\frac{1}{2}$?

7. There is a rectangular field whose breadth is $\frac{3}{5}$ of its length. After laying out $\frac{1}{10}$ of the field for a garden, there remain 54 square rods. Find the length and the breadth of the field.

8. Two young men were conversing about their ages. The younger said, "I am 25 years of age." "Then," said the older, "the sum of our ages multiplied by the difference equals 51." What was the age of the older man?

AFFECTED QUADRATIC EQUATIONS.

167. 1. What is the square of $x + 1$? Of $x + 2$? Of $x + 3$?

2. How may the first term of a binomial be found from its square?

3. Since the second term of the square of a binomial is twice the product of the terms, how may the second term of the binomial be found when the first term of the binomial is known?

4. Add a quantity to $x^2 + 2x$ that will make it a perfect square. How is the term found?

5. Add a quantity to $x^2 + 10x$ that will make it a perfect square. How is the term found?

168. A quadratic equation which contains both the first and second powers of an unknown quantity is called an **Affected Quadratic Equation**.

Thus, $x^2 + 2x = 8$, $8x^2 + 5x = 16$, and $ax^2 + bx = c$ are affected quadratic equations.

169. First method of completing the square.

1. Given $x^2 + 8x = 20$, to find the value of x .

PROCESS.

$$\begin{aligned} x^2 + 8x &= 20 \\ x^2 + 8x + 16 &= 20 + 16 \\ x^2 + 8x + 16 &= 36 \\ x + 4 &= \pm 6 \\ x &= 6 - 4 \text{ or } -6 - 4 \\ x &= 2 \text{ or } -10. \end{aligned}$$

EXPLANATION. The first member of the equation will be made a perfect square by adding the square of one half the coefficient of x to both members. One half of 8 is 4 and the square of 4 is 16. This added to each member gives $x^2 + 8x + 16 = 36$. Extracting the square root of each member, we have $x + 4 = \pm 6$. Using the first value of x , $x = 2$; using the second value of x , $x = -10$.

2. Given $x^2 + 3x = 28$, to find the value of x .

SOLUTION.	$x^2 + 3x = 28$
Completing the square,	$x^2 + 3x + \frac{9}{4} = 28 + \frac{9}{4}$, or $\frac{121}{4}$
Extracting the square root,	$x + \frac{3}{2} = \pm \frac{11}{2}$
Transposing,	$x = \frac{5}{2} \text{ or } 4$
	$x = -\frac{11}{2} \text{ or } -7$

3. Given $3x^2 + 4x = 20$, to find the value of x .

SOLUTION.	$3x^2 + 4x = 20$
Dividing by the coefficient of x^2 ,	$x^2 + \frac{4}{3}x = \frac{20}{3}$
Completing the square,	$x^2 + \frac{4}{3}x + \frac{4}{9} = \frac{20}{3} + \frac{4}{9}$, or $\frac{64}{9}$
Extracting the square root,	$x + \frac{2}{3} = \pm \frac{8}{3}$
	$\therefore x = 2 \text{ or } -\frac{10}{3}$

Find the values of x in the following equations :

4. $x^2 + 4x = 12.$

16. $x^2 - 7x = -6.$

5. $x^2 - 2x = 15.$

17. $x^2 + 10x = 56.$

6. $x^2 + 8x = 33.$

18. $x^2 + 12x = 13.$

7. $x^2 + 6x = 40.$

19. $x^2 - 30x = 64.$

8. $x^2 - 6x = 7.$

20. $x^2 + 14x = 32.$

9. $x^2 + 10x = 11.$

21. $3x^2 + 6x = 504.$

10. $x^2 - 8x = 20.$

22. $x^2 - 9x = -8.$

11. $x^2 + 20x = 125.$

23. $x^2 - 28x = 60.$

12. $3x^2 + 6x = 24.$

24. $x^2 + 11x = 80.$

13. $x^2 + 5x = 14.$

25. $x^2 - 15x = -36.$

14. $2x^2 + 5x = 18.$

26. $x^2 + 2x = 99.$

15. $3x^2 - 25x = -50.$

27. $x^2 - 3x = 4.$

170. Other methods of completing the square.

In the previous examples, when the square of the unknown quantity had a coefficient, the equation was divided by that coefficient so that the *term containing the square of the unknown quantity might be a perfect square.*

The coefficient of the second power may always be made a square, all fractions avoided, the square completed, and the values of the unknown quantity found as follows :

RULE. *Multiply the equation by four times the coefficient of the highest power of the unknown quantity ; add to each member the square of the coefficient of the first power of the unknown quantity, and then find the value of the unknown quantity by extracting the square root, etc.*

1. Solve the equation $3x^2 + 5x = 8$.

SOLUTION.

$$3x^2 + 5x = 8$$

Multiplying by 12, etc., $36x^2 + 60x + 25 = 96 + 25$ or 121

$$6x + 5 = \pm 11$$

$$6x = 6 \text{ or } -16$$

$$x = 1 \text{ or } -\frac{8}{3}$$

2. Solve the equation $ax^2 + bx = c$.

SOLUTION.

$$ax^2 + bx = c$$

Multiplying by $4a$, etc., $4a^2x^2 + 4abx + b^2 = 4ac + b^2$

$$2ax + b = \pm \sqrt{4ac + b^2}$$

$$2ax = -b \pm \sqrt{4ac + b^2}$$

$$x = -\frac{b}{2a} \pm \frac{1}{2a} \sqrt{4ac + b^2}$$

Solve the following equations:

3. $3x^2 - 4x = 55$.

14. $x^2 - 2bx = 3b^2$.

4. $2x^2 + x = 21$.

15. $x^2 + 3ax = 10a^2$.

5. $4x^2 + 3x = 7$.

16. $6x^2 - 5x = 76$.

6. $2x^2 - 9x = -4$.

17. $3x^2 - x = 70$.

7. $4x^2 + 2x = 110$.

18. $3x - \frac{169}{x} = 26$.

8. $2x^2 - 6x = 56$.

9. $5x^2 - 4x = 288$.

19. $2x - \frac{18}{x} = 9$.

10. $8x^2 - 6x = 2$.

11. $7x^2 - 4x = 20$.

20. $x(2x - 4) = x^2 + 12$.

12. $x^2 + 17x - 18 = 0$.

21. $ax^2 - 4x = a + 4$.

13. $2x^2 - 18x = -40$.

22. $2ax^2 - bx = 2a + b$.

23. $x(4x + 3) = x^2 + 90$.

24. $(x - 4)(x + 6) = 4x + 56$.

$$25. \quad 2x^2 - 5cx = -2c^2.$$

$$26. \quad x^2 - 2ax = m^2 - a^2.$$

$$27. \quad (x + a)^2 = 5ax - (x - a)^2.$$

PROBLEMS.

171. 1. Find two numbers whose sum is 10 and whose product is 24.

SOLUTION. Let $x =$ one number.

Then $10 - x =$ the other.

$$10x - x^2 = 24$$

$$x^2 - 10x = -24$$

$$x^2 - 10x + 25 = 1$$

$$x - 5 = \pm 1$$

$$x = 6 \text{ or } 4$$

$$10 - x = 4 \text{ or } 6$$

2. Divide 13 into two such parts that their product may be 36.

3. The difference between two numbers is 6, and their product is 160. What are the numbers?

4. The difference between two numbers is 4, and the sum of their squares is 40. What are the numbers?

5. An orchard containing 2808 trees had 2 trees more in a row than the number of rows. How many rows were there? How many trees were there in a row?

6. A rectangular piece of ground is 5 rods longer than it is wide and contains 500 square rods. What is the length of its sides?

7. If four times the square of a certain number is diminished by twice the number, there is a remainder of 30. What is the number?

8. The difference between two numbers is 4, and the sum of their squares is 346. What are the numbers?

9. If 6 times the square of a certain number is diminished by 10 times the number, the result will be 304. What is the number?

10. A sum of \$ 224 is divided among a certain number of men in such a manner that each man receives \$ 20 more than the number of men in the company. How many men are there?

11. There is a certain number which, being subtracted from 20 and the remainder multiplied by the number, gives a product of 100. What is the number?

12. A certain number of men pay a bill of \$ 240. If each man pays \$ 1 more than the number of men, what amount does each man pay?

13. A lady paid \$ 40 for some broadcloth. If the price per yard was \$ 6 less than the number of yards she bought, how many yards did she buy?

14. A man bought some cattle for \$ 100. If the number had been one more, the price per head would have been \$ 5 less. How many cattle did he buy?

15. A merchant sold goods for \$ 150, gaining a per cent equal to one half the number of dollars which the goods cost him. What was the cost of the goods?

SUGGESTION. Let $2x$ = the cost; then, $\frac{x}{100}$ = the gain per cent, and $\frac{2x^2}{100}$ = the gain.

16. A man sold a quantity of goods for \$ 39 and gained a per cent equal to the number of dollars which the goods cost him. How much did they cost?

SIMULTANEOUS QUADRATIC EQUATIONS.

172. The following solutions will illustrate some methods of solving **Simultaneous Quadratic Equations.**

1. Given $\begin{cases} x + y = 5 \\ xy = 6 \end{cases}$ to find the values of x and y .

SOLUTION.	$x + y = 5$	(1)
	$xy = 6$	(2)
(1) squared,	$x^2 + 2xy + y^2 = 25$	(3)
(2) $\times 4$,	$4xy = 24$	(4)
(3) $-$ (4),	$x^2 - 2xy + y^2 = 1$	(5)
Square root of (5),	$x - y = \pm 1$	(6)
(1) $+$ (6),	$2x = 6$ or 4	(7)
	$x = 3$ or 2	(8)
Substituting in (1),	$y = 2$ or 3	(9)

2. Given $\begin{cases} x + y = 10 \\ x^2 + y^2 = 52 \end{cases}$ to find the values of x and y .

SOLUTION.	$x + y = 10$	(1)
	$x^2 + y^2 = 52$	(2)
Squaring (1),	$x^2 + 2xy + y^2 = 100$	(3)
(3) $-$ (2),	$2xy = 48$	(4)
(2) $-$ (4),	$x^2 - 2xy + y^2 = 4$	(5)
Square root of (5),	$x - y = \pm 2$	(6)
(6) $+$ (1),	$2x = 12$ or 8	(7)
	$x = 6$ or 4	(8)
Substituting in (1),	$y = 4$ or 6	(9)

3. Given $\begin{cases} x + y = 5 \\ 2x^2 + y^2 = 17 \end{cases}$ to find the values of x and y .

SOLUTION.	$x + y = 5$	(1)
	$2x^2 + y^2 = 17$	(2)
From (1),	$x = 5 - y$	(3)

$$\text{Hence,} \quad 2x^2 = 50 - 20y + 2y^2 \quad (4)$$

$$\text{Substituting (4) in (2),} \quad 50 - 20y + 2y^2 + y^2 = 17 \quad (5)$$

$$\text{Collecting terms,} \quad 3y^2 - 20y = -33 \quad (6)$$

$$\text{Completing the square, etc.,} \quad y = 3 \text{ or } 3\frac{1}{3} \quad (7)$$

$$\text{Substituting (7) in (1)} \quad x = 2 \text{ or } 1\frac{1}{3} \quad (8)$$

$$4. \text{ Given } \begin{cases} x + y = 11 \\ x^2 - y^2 = 11 \end{cases} \text{ to find the values of } x \text{ and } y.$$

$$\text{SOLUTION.} \quad x + y = 11 \quad (1)$$

$$x^2 - y^2 = 11 \quad (2)$$

$$(2) \div (1) \text{ (Ax. 5),} \quad x - y = 1 \quad (3)$$

$$(1) + (3), \quad 2x = 12 \quad (4)$$

$$x = 6 \quad (5)$$

$$\text{Substituting (5) in (1),} \quad y = 5 \quad (6)$$

Find the values of the unknown quantities in the following equations:

$$5. \quad \begin{cases} x - y = 1 \\ x^2 - y^2 = 3 \end{cases} \quad 11. \quad \begin{cases} 2x + 2y = 26 \\ x^2 - y^2 = 39 \end{cases}$$

$$6. \quad \begin{cases} x + y = 20 \\ x^2 - y^2 = 120 \end{cases} \quad 12. \quad \begin{cases} 3x + y = 18 \\ x^2 + 2y^2 = 43 \end{cases}$$

$$7. \quad \begin{cases} 4x + 4y = 5 \\ 8xy = 2 \end{cases} \quad 13. \quad \begin{cases} x + y = 11 \\ x^2 - y^2 = 33 \end{cases}$$

$$8. \quad \begin{cases} x + y = 8 \\ x^2 + y^2 = 32 \end{cases} \quad 14. \quad \begin{cases} x^2 + y^2 = 29 \\ xy = 10 \end{cases}$$

$$9. \quad \begin{cases} 2x + y = 11 \\ 2x^2 + y^2 = 57 \end{cases} \quad 15. \quad \begin{cases} x^2 - y^2 = 102 \\ x - y = 3 \end{cases}$$

$$10. \quad \begin{cases} x + y = 5 \\ x^2 - 2xy - y^2 = 7 \end{cases} \quad 16. \quad \begin{cases} x + y = 12 \\ xy = 20 \end{cases}$$

$$17. \begin{cases} x - y = 1 \\ x^2 - xy + y^2 = 21 \end{cases}$$

$$24. \begin{cases} x + 2y = 9 \\ x^2 + 2y^2 = 33 \end{cases}$$

$$18. \begin{cases} x^2 - y^2 = 20 \\ x + y = 10 \end{cases}$$

$$25. \begin{cases} x - y = 7 \\ x^2 + xy + y^2 = 13 \end{cases}$$

$$19. \begin{cases} 2x + 3y = 16 \\ 2x^2 - y^2 = 46 \end{cases}$$

$$26. \begin{cases} x - y = 9 \\ xy + 8 = 0 \end{cases}$$

$$20. \begin{cases} x - y = 4 \\ 2xy = 10 \end{cases}$$

$$27. \begin{cases} x + y = a \\ xy = \frac{a^2}{4} - b^2 \end{cases}$$

$$21. \begin{cases} xy = 90 \\ x^2 + y^2 = 181 \end{cases}$$

$$28. \begin{cases} \frac{x}{a} + \frac{y}{b} = 3 \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 6 \end{cases}$$

$$22. \begin{cases} x - y = -3 \\ 4x^2 - 2y^2 = 92 \end{cases}$$

$$23. \begin{cases} 9x - 2y = 8 \\ xy = 10 \end{cases}$$

PROBLEMS.

173. 1. The sum of two numbers is 16, and their product is 63. What are the numbers?

2. The sum of two numbers is 11, and the difference of their squares is 55. What are the numbers?

3. A is 4 years older than B, and the sum of the squares of their ages is 976. What are their ages?

4. The distance around a rectangular picture is 56 inches, and its surface is 192 square inches. Find the length and the breadth of the picture.

5. There are two unequal square fields which together require 100 rods of fence to inclose them. If the sum of their areas is 325 square rods, what is the length of a side of each?

6. The difference between two numbers is 10, and the difference between their squares is 340. What are the numbers?

7. The sum of two numbers is 7, and the sum of their squares is 29. What are the numbers?

8. A and B travel a certain distance in 3 days. A travels 10 miles a day more than B, and the square of the distance B travels per day subtracted from the square of the distance A travels per day is 500 miles. What is the entire distance, and how much of it does each travel?

9. A merchant received \$10 for a certain number of yards of linen, and \$9 for 20 yards more of cotton at 10 cents less per yard. How many yards did the merchant sell of each?

10. The area of a rectangular field is 1350 square rods. If its length and breadth were each lessened 5 rods, the area would be 1000 square rods. Find the length and the breadth?

11. The sum of two numbers multiplied by their difference equals 36, and the sum of the numbers is 18. What are the two numbers?

12. A merchant has 200 yards of silk and velvet. If 50 times the number of yards of silk is subtracted from the square of the number of yards of velvet, the remainder is 400. How many yards are there of each?

13. A rectangular and a square field joined each other. A side of the square field was half the length of the rectangular field, but less than its width, and the area of both fields was $2\frac{1}{2}$ acres. What were the dimensions of each field, if it required 90 rods of fence to inclose them as one field?

GENERAL REVIEW.



I.

174. 1. When $a = 1$, $b = 2$, $c = 4$, $d = 6$, what is the value of $\frac{2a + b^2 - ab + 2c}{d - c + ab} - \frac{cd - bc}{cd - b}$?
2. Find the sum of $3a(x - y)$, $4a(x - y)$, $2b(x - y)$, and $5(x - y)$.
3. Simplify $(x + 5) - (x + 10) - [x - (3x + 25) - 10]$.
4. Add $7\sqrt{x} - 8x + 7x^2$, $-6\sqrt{x} + 4x^2 - 8x$, $7x^2 - 8\sqrt{x}$.
5. Add $ax(a - 1) + (b^2 - 2) + y^2$, $2(b^2 - 2) - 3y^2 + 3ax(a - 1)$, $5y^2 - 6ax(a - 1) - 6(b^2 - 2)$, and subtract from the result $4ax(a - 1) + (b^2 - 2) - 7y^2$.
6. What number added to three times itself equals ab ?
7. A man had property costing $4b$ dollars, $7c$ dollars, $5bc$ dollars, which he sold for \$600. How much did he gain?
8. From $205x^2 - 74y^2 + 89c^2$ subtract $35x^2 - 4y^2 - c^2 + 15$.
9. Simplify $x^2 + y^2 - \{x^2 - y^2 - (2x^2 - 4y^2) - 2z\}$.
10. What number subtracted from a times itself equals $a^2 - 1$?
11. Multiply $x^4 + 2x^3y + 4x^2y^2 + 8xy^3 + 16y^4$ by $x - 2y$.
12. Expand $(x + 3)(x - 4)(x + 4)(x - 3)(x - 1)(x + 1)$.

13. Factor $5x^2y - 10axy + 25xy^3 - 15a^2x^2y^2$.

14. Factor $x^2 + x - 156$, $x^2 - 15x + 56$, $x^2 - 3x - 70$.

15. If 7 quarts of milk cost 1 cent less than a cents, what will b quarts of milk cost?

16. With a dollars a man paid for 10 bushels of potatoes at c dollars per bushel, and received 2 dollars in change. Express this as an equation.

17. Divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$.

18. Factor $a^3 - b^6$, $125x^3 - 64y^3$, $x^4 - 81$.

19. Divide $2x^6 - 9x^4 - 8x^3 - 1$ by $x^3 + 3x^2 + 3x + 1$.

20. Find the value of x in the equation $bcx - ab = -dx - 1$.

21. Factor $4a^2 + 36a + 81$, $100x^2 - 25y^2$, $9x^2 - 42xy + 49y^2$.

22. A merchant sold 6 pounds of coffee at b cents a pound, 5 pounds at c cents a pound, and b pounds at a cents a pound. What was the average price received per pound for the coffee?

23. Find the value of x in the equation $ax - a^2 - b^2 = 2ab - bx$.

24. Factor $8x^3 + 512$, $a^2b^2 - 1$, $m^3 + 27$.

25. Solve $(x - 5)(x + 4) = x(x - 5) + 8$.

II.

26. Write out the following squares:

$$(x + 2y)^2, (4a - 3b)^2, (2m - 5)^2, (3ab + 1)^2, (5 + 2a)^2, \\ (6x - 7y)^2, (a^2 + 10)^2, (m^2 - 4n^2)^2, (x^2 - 3y)^2, (3a + 5x)^2.$$

27. Write out the following products :

$$(x+7)(x-4), \quad (x+15)(x-15), \quad (x+10)(x+3), \\ (x-12)(x+9), \quad (2x+5)(2x-5), \quad (m^2+1)(m^2-1), \\ (x-8)(x-11), \quad (x+15)(x-14), \quad (3x+7)(3x-7).$$

28. Find the highest common divisor of

$$3a^5 - 48a, \quad 2a^3b - 16b, \quad \text{and} \quad 5a^2c - 20c.$$

29. Find the highest common divisor of

$$x^2 + 6x + 9, \quad x^2 + x - 6, \quad 3x^2 + 7x - 6.$$

30. Find the lowest common multiple of

$$x^2 + 2x - 3, \quad x^2 - 3x + 2, \quad x^2 + x - 6.$$

31. Find the lowest common multiple of

$$x^2 - 9, \quad x^2 + 9x + 18, \quad x^2 + 3x - 18.$$

32. Reduce $\frac{a^2 + ab + b^2}{a^3 - b^3}$ to its lowest terms.

33. Reduce $\frac{x^2 - 4y^2}{x^2 - 3xy - 10y^2}$ to its lowest terms.

34. Simplify $\left(\frac{4}{x+y} + \frac{2}{x-y}\right) \div \frac{3x-y}{x+y}$.

35. Simplify $\frac{5}{x+3} \times \left(\frac{2x^2+12}{5} - \frac{x^2+9}{3}\right)$.

36. Simplify $\frac{1}{4(1+y)} - \frac{1}{4(y-1)} + \frac{2}{2(y^2-1)}$.

37. Simplify $\frac{x(a-x)}{a^2+2ax+x^2} \times \frac{x(a+x)}{a^2-2ax+x^2}$.

38. Simplify $\frac{a^2-x^2}{a+b} \div \frac{ax+x^2}{a^2-b^2}$.

39. Simplify $\frac{3a-4b}{7} - \frac{2a-b-c}{3} + \frac{15a-4c}{12}$.

40. A can do a piece of work in m days; B can do it in n days. Express the part that each can do in one day. Express the part that both can do in one day, and the number of days in which both can do the work.

41. Solve $\frac{3x+5}{7} + 10 = \frac{3x}{5} + \frac{2x+7}{3}$.

42. Solve $x(x-3) + \frac{x}{6} = x(x-5) + \frac{13}{3}$.

43. Solve $\frac{x-2}{x^2-4} = \frac{x+2}{x^2-4} - \frac{4}{x+2}$.

44. Solve $\frac{a(a-x)}{b} - \frac{b(b+x)}{a} = x$.

45. A farmer has one third as many horses as cows, and one half as many cows as sheep. If there are a animals in all, how many are there of each kind?

46. Solve $\frac{a}{x-a} - \frac{b}{x-b} = \frac{a-b}{x-c}$.

47. Solve $\frac{5}{1-5x} + \frac{4}{2x-1} = \frac{3}{3x-1}$.

48. The sum of two numbers is a , and the first divided by m equals the second divided by n . What are the numbers?

49. The m th part of a certain number plus 10 equals m times the number minus 5. What is the number?

50. If a certain number is divided successively by a , b , and c , the sum of the quotients will be 10. What is the number?

III.

Solve the following equations:

$$51. \begin{cases} 4x + 3y = 3 \\ 10x - 9y = 2 \end{cases}$$

$$52. \begin{cases} x + \frac{y+1}{4} = 6 \\ \frac{x+1}{4} + y = 15 \end{cases}$$

$$53. \begin{cases} \frac{x-2}{5} - \frac{10-x}{3} = \frac{y-10}{4} \\ \frac{2y+4}{3} - \frac{2x+y}{8} = \frac{x+13}{4} \end{cases}$$

$$54. \begin{cases} \frac{1}{x} + \frac{3}{y} = a \\ \frac{5}{x} + \frac{2}{y} = b \end{cases}$$

$$56. \begin{cases} \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 62 \\ \frac{x}{4} + \frac{y}{5} - \frac{z}{12} = 8 \\ \frac{x}{3} - \frac{y}{4} + \frac{z}{5} = 17 \end{cases}$$

$$55. \begin{cases} 4x + 12y = 5 \\ 6x - 3z = 2 \\ 16x - 9z = 1 \end{cases}$$

$$57. \begin{cases} x + y = a \\ x + z = b \\ y + z = c \end{cases}$$

$$58. \frac{x^2}{5} - \frac{x^2 - 10}{15} = 7 - \frac{50 + x^2}{25}.$$

$$59. \frac{a-2}{1+x} = \frac{1-x}{ax^2}.$$

$$62. x^2 + 15x = 34.$$

$$63. 4x^2 - x = 33.$$

$$60. \frac{x+4}{x-4} + \frac{x-4}{x+4} = \frac{10}{3}.$$

$$64. x^2 - 2ax = m - a^2.$$

$$65. 2ax^2 - 5x = 2a + 5.$$

$$61. x^2 - 3x = 4.$$

$$66. 5x^2 + 4x = 9.$$

Solve by factoring:

$$67. x^2 - 5x - 104 = 0.$$

$$68. x^2 - 18x + 72 = 0.$$

$$69. x^2 - 12x + 30 = 58.$$

70. What quantity multiplied by itself gives $x^2 + 4xy - 6x + 4y^2 - 12y + 9$?

71. Extract the square root of 60,025.

Find the values of x and y in the following equations:

$$72. \begin{cases} x + 2y = 11 \\ 2x^2 + y^2 = 59 \end{cases}$$

$$74. \begin{cases} x^2 - y^2 = 36 \\ x - y = 2 \end{cases}$$

$$73. \begin{cases} x^2 + y^2 = 37 \\ 3xy = 18 \end{cases}$$

$$75. \begin{cases} x^2 + 2xy - y^2 = 73 \\ x - y = 5 \end{cases}$$

IV.

76. One half of Tom's money equals $\frac{4}{7}$ of John's, and Tom has \$12 more than John. How much money has each?

77. At a certain election there were two candidates; the successful candidate had a majority of 60, which was $\frac{1}{20}$ of all the votes cast. How many votes did the defeated candidate receive?

78. A lad spent on July 4th $\frac{1}{2}$ of his money and 6 cents more for firecrackers, and $\frac{1}{3}$ of his money and 4 cents more for torpedoes. If that was all of his money, how much had he?

79. A man bought a number of sheep for \$225; 10 of them having died, he sold $\frac{4}{5}$ of the remainder at cost for \$150. How many sheep did he buy?

80. If to the numerator of a certain fraction 1 is added, the value of the fraction becomes 1; but if 3 is subtracted from the denominator, the value becomes 2. What is the fraction?

SUGGESTION. Let $\frac{x}{y}$ represent the fraction.

81. A man bought 20 bushels of wheat and 15 bushels of corn for \$36, and 15 bushels of wheat and 25 bushels of corn, at the same rate, for \$32.50. What did he pay per bushel for each?

82. The sides of a rectangular court are to each other as 3 to 4, and the surface is 2700 square feet. What are the lengths of the sides?

83. If $\frac{2}{3}$ of the value of a carriage is equal to $\frac{3}{4}$ of the value of a horse, and the value of the carriage is \$20 more than the value of the horse, what is the value of each?

84. The sum of two numbers is 34, and their product is 285. What are the numbers?

85. A merchant, after selling from a cask of vinegar 15 gallons more than $\frac{1}{7}$ of the whole, found that he had left just 4 times as much as he had sold. How many gallons did the cask contain at first?

86. The difference between two numbers is 6, and one half of their product equals the cube of the smaller. What are the numbers?

87. A company of 20 men and women paid a bill of \$75. The men paid $\frac{4}{5}$ of the bill, and by so doing each man paid \$1 more than each woman. How many men were there in the company? How many women were there?

88. A and B own flocks of sheep. If A sells to B 10 sheep, they will each have an equal number; but if B sells to A 10 sheep, A will have three times as many as B. How many sheep has each?

89. What are the two numbers whose difference is 8, and whose sum multiplied by their difference is 240?

SUGGESTION. Let $x + 4$ and $x - 4$ equal the numbers.

90. A grocer will sell 1 pound of tea, 2 pounds of coffee, and 1 pound of sugar for \$1.00; or he will sell 2 pounds of tea and 6 pounds of sugar for \$1.00; or he will sell 3 pounds of coffee and 2 pounds of sugar for \$1.00. What is the price per pound of each?

91. How far may a person ride in a stage, going at the rate of 8 miles an hour, if he is gone 11 hours, and walks back at the rate of 3 miles an hour?

92. I have two purses, one containing gold coins, and the other silver coins. The money in both purses equals in value twice the value of the gold coins; and $\frac{1}{2}$ the value of the silver coins, increased by $\frac{1}{2}$ the value of the gold coins, equals \$150. How much does each purse contain?

93. A man bought a horse, a cow, and a sheep for a certain sum. The horse and the sheep cost 5 times as much as the cow, and the sheep and the cow cost $\frac{2}{7}$ as much as the horse. How much did each cost, if the cow cost \$30?

94. Find two numbers, such that their product is 21, and their product added to the sum of their squares is 79.

95. A merchant sold some goods for \$96, thereby gaining as much per cent as the goods cost. What was the cost of the goods?

SUGGESTION. Let x = the cost; then $\frac{x}{100}$ = the gain per cent.

96. A and B have the same income. A saves $\frac{1}{6}$ of his, but B, by spending \$100 each year more than A, at the end of 5 years finds himself \$240 in debt. What is the income of each?

97. A farmer had his sheep in 3 fields. $\frac{2}{3}$ of the number in the first field was equal to $\frac{3}{4}$ of the number in the second field, and $\frac{2}{3}$ of the number in the second field was $\frac{3}{4}$ of the number in the third field. If the entire number was 434, how many were there in each field?

98. A man left one half of his property to his wife, one sixth to his daughter, one twelfth to his brother, and the rest, which was \$900, to charitable institutions. How much was his property worth?

99. If the length and the breadth of a rectangle were each increased by 1, the area would be 120; but if they were each diminished by 1, the area would be 80. Find the length and the breadth.

100. A grass plot 9 yards long and 6 yards broad has a path around it. If the area of the path equals the area of the plot, what is the width of the path?

101. A and B traveled toward each other from two towns 247 miles apart. A went 9 miles per day, and the number of days before they met was 3 more than the number of miles B went per day. How far did each travel?

102. Three equal square lots have an area of 193 square rods less than another square lot whose sides are each 13 rods longer than the sides of each of the three equal square lots. Find the length of a side of each.

103. A rectangular plot of ground is surrounded by a street of uniform width. If the plot is 50 rods long and 40 rods wide, and the area of the street is 784 square rods, what is the width of the street?

V.

104. Find the highest common divisor of $a^3 + x^3$, $(a + x)^3$, and $a^2 - x^2$.

105. Find the highest common divisor of $9x^2 - 1$, $9x + 3$, and $(3x + 1)^2$.

106. Find the highest common divisor of $x^2 - 2x - 3$, $x^2 + 4x + 3$, and $x^2 - 5x - 6$.

107. Find the lowest common multiple of $3(x^2 + xy)$, $5(xy - y^2)$, and $6(x^2 - y^2)$.

108. Find the lowest common multiple of $x^2 + 11x + 30$, $x^2 + 12x + 35$, and $x^2 + 13x + 42$.

109. Find the lowest common multiple of $12xy(x^2 - y^2)$, $2x(x + y)^2$, and $3y(x - y)^2$.

110. Reduce $\frac{3ab(a^2 - b^2)}{4(a^2b - ab^2)^2}$ to its lowest terms.

111. Reduce $\frac{a^3 + 1}{a^3 + 3a^2 + 3a + 1}$ to its lowest terms.

Change to equivalent fractions having their lowest common denominator:

112. $\frac{8x + 2}{x - 2}$, $\frac{2x - 1}{3x - 6}$, and $\frac{3x + 2}{5x - 10}$.

113. $\frac{x - 3}{x^2 - 4}$, $\frac{x - 2}{x^2 + x - 6}$, and $\frac{x^2 + 4}{x^2 + 6x + 9}$.

114. $\frac{x}{x + y}$, $\frac{xy}{x^2 - y^2}$, and $\frac{y^2}{x^2 + y^2}$.

115. $\frac{3}{1 + x}$, $\frac{5}{4 + 4x}$, and $\frac{2x}{1 - x^2}$.

Simplify :

$$116. \frac{a}{a-x} - \frac{x}{a+2x} - \frac{a^2+x^2}{(a-x)(a+2x)}$$

$$117. \frac{x+y}{y} - \frac{2x}{x+y} + \frac{x^2y-x^3}{y(x^2-y^2)}$$

Solve :

$$118. \frac{x-3}{4(x-1)} = \frac{x-5}{6(x-1)} + \frac{1}{9}$$

$$119. \frac{7}{x+1} = \frac{6x}{x-1} - \frac{3(1+2x^2)}{x^2-1}$$

$$120. \frac{17}{5(x+3)} + \frac{4}{5} = \frac{21+2x}{3(x+3)} - 2.$$

$$121. \frac{3x-1}{2x-1} - \frac{4x-2}{3x-2} = \frac{1}{6}$$

$$122. \frac{2}{2x-3} = \frac{6}{3x+2} - \frac{1}{x-2}$$

$$123. am - b + \frac{x}{m} = \frac{ax}{b}$$

$$124. \frac{3ax-2b}{3b} - \frac{ax-a}{2b} = \frac{ax}{b} - \frac{2}{3}$$

$$125. \frac{ab}{x} = bc + d + \frac{1}{x}$$

$$126. \frac{2}{ac} + \frac{2}{ab} + \frac{4}{bc} = 2ax + bx + cx.$$

$$127. \frac{1}{2}\left(x - \frac{a}{3}\right) + \frac{1}{4}\left(x - \frac{a}{5}\right) = \frac{1}{3}\left(x - \frac{a}{4}\right).$$

$$128. \frac{x^2-x+1}{x-1} - \frac{x^2+x+1}{x+1} = \frac{2x+1}{x^2-1}$$

VI.

Solve:

$$129. \begin{cases} 2x - 3y + z = 1 \\ 4x + 2y - 3z = 13 \\ 3x - 5y + 4z = 3 \end{cases}$$

$$131. \begin{cases} \frac{x-a}{b} = \frac{b-y}{a} \\ x + \frac{y-b}{a} = \frac{a+y-x}{b} \end{cases}$$

$$130. \begin{cases} \frac{x}{2} + \frac{y}{4} + \frac{z}{6} = 3 \\ \frac{x}{5} + y + \frac{z}{10} = 5 \\ \frac{x}{4} + y - \frac{z}{12} = 4 \end{cases}$$

$$132. \begin{cases} ax + by = c^2 \\ \frac{a}{b+y} = \frac{b}{a+x} \end{cases}$$

$$133. \begin{cases} \frac{x-a}{y-b} = c \\ a(x-a) + b(y-b) = -abc \end{cases}$$

$$134. \begin{cases} \frac{x}{a+b} + \frac{y}{a-b} = 2a \\ \frac{x-y}{4ab} = 1 \end{cases}$$

Find the square root of the following:

135. 8836.

139. .00103041.

136. 5625.

140. .00450241.

137. 106929.

141. 4782969.

138. 10.4976.

142. 43046721.

143. $4x^4 - 4x^3 + 13x^2 - 6x + 9.$

144. $16x^4 + 24x^3 + 89x^2 + 60x + 100.$

145. $x^2 + 2xy + y^2 + 6xz + 6yz + 9z^2.$

146. $z^2 - 2z + 1 + 2mz - 2m + m^2.$

147. $x^6 - 6x^5 + 17x^4 - 34x^3 + 46x^2 - 40x + 25.$

Find the cube root of the following :

148. 356400829.

151. 820.025856.

149. 521660125.

152. 8653.002877.

150. 102503.232.

153. 1.371330631.

154. $8x^6 + 48ax^5 + 60a^2x^4 - 80a^3x^3 - 90a^4x^2 + 108a^5x - 27a^6$.

155. $27m^6 - 108m^5 - 45m^4 + 440m^3 + 105m^2 - 588m - 343$.

156. $a^9 - 3a^8 + 8a^6 - 6a^5 - 6a^4 + 8a^3 - 3a + 1$.

VII.

Solve :

157. $\begin{cases} xy = 7 \\ x^2 + y^2 = 50 \end{cases}$

162. $\begin{cases} \frac{1}{x} + \frac{1}{y} = 6 \\ \frac{1}{x^2} - \frac{1}{y^2} = 12 \end{cases}$

158. $\begin{cases} xy = 12 \\ x - 2y = 5 \end{cases}$

159. $\begin{cases} x - y = 1 \\ x^2 + xy + y^2 = 19 \end{cases}$

163. $\begin{cases} \frac{x}{a} + \frac{y}{b} = 1 \\ \frac{a}{x} + \frac{b}{y} = 4 \end{cases}$

160. $\begin{cases} x - y = a \\ x^2 - y^2 = ab \end{cases}$

161. $\begin{cases} x - y = 2 \\ x^2 - y^2 = 20 \end{cases}$

164. $\begin{cases} x^2 + xy + y^2 = 37 \\ x^2 - xy + y^2 = 13 \end{cases}$

165. The head of a fish is 8 inches long. The tail is as long as the head and $\frac{1}{2}$ of the body, and the body is as long as the head and tail. What is the length of the fish ?

166. A tree is broken into three pieces. The part standing is 8 feet long. The top piece is as long as the part standing and $\frac{1}{4}$ of the middle piece, and the middle piece is twice as long as the other pieces. How high was the tree ?

167. A person, being asked the time of day, replied that it was past noon, and that $\frac{1}{2}$ of the time past noon was equal to $\frac{1}{4}$ of the time to midnight. What was the time?

168. A yacht, whose rate of sailing in still water is 12 miles an hour, sails down a river whose current is 4 miles an hour. How far may it go, if it is to be gone 15 hours?

169. Three men engage to husk a field of corn. The first can do it in 10 days, the second in 12, and the third in 15 days. In what time can they do it together?

170. Fifteen persons agree to purchase a tract of land, but three of the company withdrawing, the investment of each of the others is increased \$150. What is the cost of the land?

171. C and D have the same income. C saves $\frac{1}{12}$ of his, but D, by spending \$65 more each year than C, at the end of 6 years finds himself \$60 in debt. How much does each spend yearly?

172. I sold a bureau to A for $\frac{1}{4}$ more than it cost me. He sold it for \$6, which was $\frac{2}{5}$ less than it cost him. What did the bureau cost me?

173. A man agreed to work 16 days for \$24 and his board, but he was to pay \$1 a day for his board every day he was idle. If he received \$14, how many days did he work?

174. A man can saw 2 cords of wood per day, or he can split 3 cords of wood after it is sawed. How much must he saw that he may be occupied the rest of the day in splitting it?

175. A carriage maker sold 2 carriages for \$300 each. Did he gain or lose by the sale, if on one he gained 25 per cent, and on the other he lost 25 per cent?

176. A steamboat makes a trip from P to H and return in 10 hours, stopping 40 minutes at H . If it goes 6 miles an hour upstream and 10 miles an hour downstream, what is the distance between P and H ?

177. How many acres are there in a square tract of land containing as many acres as there are boards in the fence inclosing it, if the boards are 11 feet long, and the fence is 4 boards high?

178. The area of a square figure will be doubled if its length is increased 6 inches and its breadth 4 inches. Find the length of the side of the square.

179. A certain iron bar weighs 36 pounds. If the bar had been 1 foot longer, each foot would have weighed $\frac{1}{2}$ a pound less. Find the length of the bar and the weight per foot.

180. A man has two horses, and a saddle worth \$10. The value of the saddle and the first horse is double that of the second horse, but the value of the saddle and the second horse lacks \$13 of being equal to the value of the first horse. What is the value of each horse?

181. How many quantities each equal to $a^2 - 2a + 1$ must be added together to produce $5a^6 - 6a^5 + 1$?

182. There is a cistern into which water is admitted by three faucets, two of which are of the same size. When they are all open the cistern will be filled in 6 hours, but if one of the equal faucets is closed the other two will require 8 hours and 20 minutes to fill it. In what time can each faucet fill the cistern?

183. A farmer purchased 100 acres of land for \$2450. For a part of it he paid \$20 an acre and for the rest \$30 an acre. How many acres were there in each part?

QUESTIONS FOR REVIEW.

175. How does the algebraic solution of a problem differ from the arithmetical solution?

What are the letters used in algebra commonly called?

Define unknown numbers or quantities.

What letters are used to represent unknown quantities?

What is an equation?

Give the sign of equality; the sign of deduction.

What is an algebraic expression? Illustrate.

How do the uses of signs in algebra differ from their uses in arithmetic?

What is a power? Give an illustration.

What is an exponent? Give an illustration.

How are powers named? What, also, is the *second* power called? What, the *third*?

What is a coefficient? When no coefficient is expressed, what is the coefficient? Give an illustration.

How should quantities inclosed in parentheses be treated?

What is an algebraic term?

Distinguish between positive and negative terms. Illustrate each.

Distinguish between similar and dissimilar terms. Illustrate each.

What is a monomial? A polynomial? A binomial? A trinomial? Give an illustration of each.

Explain what is meant by using the signs $+$ and $-$ as signs of opposition.

When positive quantities are added, what is the sign of the sum?

What is the sign of the sum when negative quantities are added?

How is the sign of the result determined when both positive and negative quantities are added?

What kind of quantities can be united by addition into one term?

How may dissimilar quantities be treated in addition?

What are the two cases in addition? Give the rule.

Define known numbers or quantities. What letters of the alphabet represent them?

Instead of subtracting a positive quantity, what may be done to secure the same result?

Instead of subtracting a negative quantity, what may be done to secure the same result?

Give the three principles in subtraction.

Give the cases in subtraction.

When is it necessary in subtraction to inclose the coefficient of the answer in parentheses?

Give the signs of aggregation. What do they show?

How may the subtrahend sometimes be expressed?

When the subtrahend is inclosed in parentheses and preceded by the sign minus, what must be done when the subtraction is performed?

Give the two principles relating to parentheses, or other signs of aggregation.

How does the sign *plus* before parentheses affect the quantity inclosed?

When a term is changed from one member of an equation to another, what change must be made in its sign?

What are the members of an equation? What is the first member? The second member? Illustrate.

What is an axiom? Give six axioms.

What is transposition? Give the principle relating to transposition. Give the rule.

How may the value of the unknown quantity obtained by solving an equation be verified? Illustrate the process by an appropriate example.

When a *positive* quantity is multiplied by a *positive* quantity, what is the sign of the product?

When a *negative* quantity is multiplied by a *positive* quantity, or a *positive* quantity by a *negative* quantity, what is the sign of the product?

When a *negative* quantity is multiplied by a *negative* quantity, what is the sign of the product?

Name the four ways of indicating multiplication.

Give the three principles of multiplication. Give the cases in multiplication. Give the rules.

What is the principle relating to the square of the sum of two quantities? What is the square of $x + y$?

What is the principle relating to the square of the difference of two quantities? What is the square of $x - y$?

What is the principle relating to the product of the sum and difference of two quantities? What is the product of $(x + y)(x - y)$?

What is the principle relating to the product of two binomials which have a common term? What is the product of $(x + 3)(x + 2)$?

What are simultaneous equations? Illustrate.

Define elimination. Give the principle relating to elimination by addition or subtraction. Give the rule.

Illustrate the method of elimination by addition or subtraction by the solution of a problem.

What is the sign of the quotient when a *positive* quantity is divided by a *positive* quantity?

What is the sign of the quotient when a *negative* quantity is divided by a *negative* quantity?

What is the sign of the quotient when a *positive* quantity is divided by a *negative* quantity, or a *negative* quantity is divided by a *positive* quantity?

How is the exponent of a quantity in the quotient found?

Give the principles relating to division. Give the cases in division. Give the rule under each case.

What is a factor? What is factoring? Give the rule for factoring a polynomial all of whose terms have a common factor. How may a polynomial be factored when only some of its terms have a common factor?

What is meant by "square root"? When can a trinomial be separated into two equal factors? Give the rule. When can a binomial be resolved into two binomial factors? Give the rule.

What is a quadratic trinomial? Illustrate the method of factoring a quadratic trinomial.

What quantity will divide the sum of two cubes? How, then, may the sum of two cubes be factored? Give the factors of $x^3 + y^3$. What quantity will divide the difference of two cubes? How, then, may the difference of two cubes be factored? Give the factors of $x^3 - y^3$.

Explain the solution of four kinds of equations by factoring.

What is a common divisor? Illustrate the use of the common divisor by giving two or more quantities and a common divisor or factor of them. What is the highest common divisor or factor of several quantities? Give the principle relating to the highest common divisor.

What is a multiple of a quantity? What is a common multiple of several quantities? What is the lowest common multiple of several quantities? Give the principle relating to the lowest common multiple. Find the lowest common multiple of two or more quantities which are monomials, and of two or more which are polynomials.

Define a fraction ; an entire quantity ; a mixed quantity. Illustrate each. What is meant by the sign of a fraction ? Give the principle relating to reduction of fractions. When is a fraction in its lowest terms ?

Name the five cases under reduction of fractions, and illustrate by solving examples. Define and illustrate similar and dissimilar fractions. What is meant by lowest common denominator ? Illustrate. Give the principles relating to the lowest common denominator. Give the rule for reducing dissimilar to similar fractions.

What effect has it upon the equality of the members of an equation to multiply both members by the same quantity ?

How may an equation with fractions be changed into one without fractions ? What is meant by clearing an equation of fractions ? Give the principle and the rule.

If a fraction has the minus sign before it, what must be done when the denominator is removed ? If a fraction is multiplied by its denominator, what is the effect ? What kind of fractions can be added or subtracted ? What must be done to dissimilar fractions before they can be added or subtracted ? Give the principles.

In what two ways may a fraction be multiplied, and in what two ways may it be divided by an entire quantity ? How should entire and mixed quantities be changed before multiplying ? How may an entire quantity be changed to a fractional form ?

When may cancellation be used in multiplication of fractions ? Solve an example in multiplication of fractions and one in division of fractions, making use of factoring and cancellation. Explain the reason for inverting the divisor in division of fractions.

How may an unknown quantity be eliminated from two simple simultaneous equations by comparison ? By substitution ? Give the rules. Illustrate by solving examples.

Define: equation; members of an equation; transposition; axiom; simultaneous equations; elimination.

What is meant by the degree of an equation? Define the term simple equation; quadratic equation. What is the degree of $x + a = b$? $x^2 + 2x = 4$? $x + xy = 6$?

Describe the method of solving, and solve a set of simultaneous equations containing three unknown quantities.

What is a power of a quantity? How many times is a quantity used as a factor in producing the second power? The third power? The n th power?

What is involution? Give the principles. Give the rule. How is a fraction raised to any power?

What is the sign of any power of a *positive* quantity? What is the sign of any *even* power of a *negative* quantity? Of any *odd* power of a *negative* quantity?

What is a root of a quantity? How are roots named?

What is the radical sign? What does it indicate? What is the index of a root?

What is evolution? Give the principles. Illustrate their application. Give the rule under each case of evolution.

How is the root of a fraction found?

Explain the method of extracting the square root of a polynomial. Give the principles relating to square root of numbers. Give the rule and solve an example.

How is the cube root of a polynomial found? Illustrate by an example. Give the principles relating to cube root of numbers. Give the rule and solve an example.

Define a quadratic equation; a pure quadratic equation. What is the root of an equation? Give an example of a pure quadratic equation and solve it. What is an affected quadratic equation? Give an example.

How many methods are given for completing the square? Give the rule for each method and solve examples.

Give four types of simultaneous quadratic equations.

ANSWERS.

Page 8. — 1. 30, John ; 10, James. 2. 20 bu. ; 40 bu. 3. \$ 2000, A ; \$ 1000, B.

Page 9. — 4. 5. 5. 30 yr., father ; 10 yr., son. 6. 15 and 60. 7. \$ 200, horse ; \$ 400, carriage. 8. 100 pear ; 200 cherry ; 400 apple. 9. 80 yr., A ; 40 yr., B. ; 20 yr., C. 10. 7 and 42. 11. 5. 12. 3, Jane ; 6, Hannah ; 30, Mary.

Page 10. — 13. 4 cents, ink ; 20 cents, pens ; 72 cents, paper. 14. \$ 6, hat ; \$ 12, trousers ; \$ 42, overcoat. 15. \$ 5, first ; \$ 10, second ; \$ 15, third ; \$ 20, fourth. 16. 20 sheep, 1st ; 40 sheep, 2d ; 60 sheep, 3d ; 80 sheep, 4th. 17. 7, 42, 70. 18. \$ 1000, A ; \$ 100, B ; \$ 400, C ; \$ 4800, D. 19. \$ 12, silver watch ; \$ 120, gold watch. 20. 360 votes, A ; 90 votes, B. 21. \$ 25, cow ; \$ 50, wagon ; \$ 100, horse.

Page 11. — 22. \$ 2000, A ; \$ 6000, B ; \$ 6000, C. 24. 6 acres, potatoes ; 30 acres, oats ; 150 acres, wheat. 25. \$ 1200. 26. 7 hens ; 14 ducks ; 98 chickens. 27. 9 and 27. 28. 9 gal. ; 36 gal.

Page 12. — 29. 700, smaller ; 5600, larger. 30. 7. 31. 5 days. 32. 11 yr., son ; 55 yr., father. 34. 15. 35. 3. 36. 7 cows ; 3 horses. 37. 4 bu. ; 14 bu.

Page 13. — 38. 24. 39. 18. 40. 18 and 27. 41. 12. 42. 20. 43. 9 marbles. 44. 12. 45. 80, 1st ; 160, 2d ; 20, 3d. 46. 6 and 30. 47. 20, 40, and 60. 48. \$ 200, A ; \$ 100, B ; \$ 600, C. 49. 126.

Page 14. — 50. 24, Henry ; 8, James. 51. 12 ¢, May ; 4 ¢, Julia ; 2 ¢, Hattie. 52. 30. 53. 500, history ; 1000, science, 400, fiction. 54. 70. 55. \$ 15, A ; \$ 9, B ; \$ 3, C. 56. 40.

Page 15. — 58. 10 apples. 59. \$ 5, 1st ; \$ 10, 2d ; \$ 35, 3d. 60. 30, 1st ; 30, 2d ; 40, 3d. 61. \$ 500. 62. 48. 64. 14. 65. 4, 16, and 14. 66. 5 and 20.

Page 16. — 67. \$ 300, 1st son ; \$ 100, 2d ; \$ 200, 3d. 68. \$ 125, 1st ; \$ 125, 2d ; \$ 62½, 3d ; \$ 187½, 4th. 69. 5 ¢, Emma ; 10 ¢, Anna. 70. 10 mi. ; 8 mi. 71. 60. 73. \$ 500, 1st month.

Page 17. — 74. 11¼ mi., stage ; 34½ mi., steamboat ; 277½ mi., railroad. 75. 12, Harry ; 4, Edward. 76. \$ 100. 77. \$ 2000, lot ; \$ 4000, house ; \$ 1000, furnishing. 78. 10 trees in each of the 5 rows ; 20 trees in each of the 8 rows. 79. \$ 3 per day ; \$ 1 per day board. 80. 40 yr. 81. \$ 3.

Page 18. — 83. 4 quarter dollars ; 8 half dollars. 84. \$ 1.25, man ; \$ 1.25, wife ; \$.25 son. 85. 30 rods, length ; 15 rods, breadth. 86. 30 ft. left standing ; 15 ft. broken off. 87. \$ 400, 1st ; \$ 200, 2d ; \$ 200, 3d. 88. \$ 2000, 1st yr. ; \$ 8000, 2d yr. ; \$ 32,000, 3d yr.

Page 22. — 1. 72. 2. 480. 3. 72. 4. 360. 5. 13. 6. 10. 7. 5. 8. 768. 9. 8. 10. 9. 11. 20. 12. 21. 13. 82. 14. 78. 15. 25. 16. 16. 17. 45. 18. 8. 19. 4. 20. 18. 21. 586. 22. 35. 23. 37. 24. 26.

Page 27. — 1. $18x$. 2. $24ab$. 3. $-20mn$. 4. $-25x^2y$. 5. $19ax$. 6. $-28bc$. 8. $10a$. 9. x^2y^2 . 10. $-mx$. 11. $5y$. 12. $6m$. 13. $8bc$. 14. $10xy$. 15. x^2z .

Page 28. — 16. $8mn$. 17. $11a^2b$. 18. 0. 19. $9(ab)^2$. 20. $7(a-b)$. 21. 0. 22. $6(x+y)$. 23. $10(a+b)^2$. 24. $6cd$. 25. $(xy)^3$. 2. $5a-7b$. 3. $8x+7y-2z$. 4. $8xy+6y^2+3z^2$.

Page 29. — 5. $4c+7d$. 6. $6n-2m$. 7. $4a+6b+c$. 8. $5x^2z+10xz^2-3xy$. 9. $5w-4y$. 10. $4a^2b^2-5c^2-cd$. 11. $2ab-1$. 12. $12a-2d$. 13. $7m+6n-x+5y$. 14. $6y+4z+2w$. 15. $-x^2y-2xy^2$. 16. $11x^m+y^n$. 17. $9a^2b^2xc-3ed$. 18. $9m^2+3mn$. 19. $22ax^3-28ay^3+9az^3$. 20. $8a^2b+8a^3bc-20b^2c^4+8$. 21. $-10x^3+17ax^2-8a^2x+9a^3$. 22. $(2+a-2b)(c+d)$. 23. $2m$.

Page 30. — 24. $10ax+11x$. 25. $5(x+y)+9z-4$. 1. $(a+b+8)$ barrels. 2. $(m+d+6)$ cents. 3. $(2c+2n)$ dollars. 4. $(a+b+3)$ miles.

Page 31. — 5. $(2c+400)$ dollars. 6. $b+1$; $b+2$. 7. $4\frac{1}{2}m$ years. 8. $(c+d+75)$ dollars. 9. $7x$; ax . 10. $(b+c+d+2)$ pounds. 11. $(m+n+p+g+r)$ dollars. 12. $(a+b+d)$ dollars. 13. $(a+b+c+25)$ dollars.

Page 34. — 3. $15x$. 4. $4ab$. 5. $5m^2$. 6. $-4xy$. 7. $-2x^2y^2z$. 8. $-5mn$. 9. $4x+4y$. 10. $5ab-c$. 11. $4b-4d$. 12. $3x+2y$. 13. $x+z$. 14. $4a+3b+c$. 15. $2a-b$. 16. $2x^2-5y^2-4xy$. 17. $6xy+2z$. 18. $(a^2+b^2)-3c^2+2d$. 19. $3a(p+q)+5$. 20. $4z-3y+1$. 21. $4x^2+y+5z-3v$. 22. $5x-3(y+z)+2$. 23. $-2a^3+b^2-3c^2+4d^2$.

Page 35. — 2. a^3x . 3. $16x^2yz$. 4. $y-6z$. 5. $3x^2+13y^2$. 6. $-6b+2c$. 7. $6ab-8c+2d$. 8. $-x^2-8xy+2x$. 9. $5ax-6y+2z^2$. 10. $8x^5-29x^3y^3+8xy$. 11. $-2ab+5b^2-8cd$. 12. $6x^2y-6xy^2-3xy-3$. 13. $3x^m+4x^my^n+4y^n$. 14. $-4(m+n^2)-4(m^2+n)$. 15. $6x^3+6y^3-3z^3+4r^3$. 16. $-3(p+q)-8(r+s)-10$.

Page 36. — 17. $20m^2nx^2+12a^2bc^2-4abd$. 18. $-a^2+c^2-6b^2$. 19. $-8xy$. 20. $x^4+2x^2y^2-2y^4$. 21. $9x^3y^3-11yz^3$. 22. $-x^m-6x^ny^m+5y^m$. 23. $5bx^3+3ay^2+9-2cy$. 24. $4m^2+mn-9n^2-22$. 25. $-2a^2x-x^3y-3$. 26. $11x^2y^2+4y^2-z^2-5$. 27. $x-y-z-u$. 28. $5a^2b^2-17bc+9b^2-4$. 30. $(a-2)x+(b+2)y$. 31. $(2c-5)x-(a+c)y+(a+3)z$.

Page 37. — 32. $(c-6)x-2aby+(4a^2-3)b^2$. 33. $(2a-3c)(x-y)+(4ab-2a)x$. 34. $(a-b)x+(b-c)y+(c-d)z$. 35. $(3-bc)x$

- + $(7 + a)y + (8 - a)z$. 36. $(5a - c)y + (2c + a)z + (6 + d)x$.
 37. $-bx + cy$. 38. $(7 - 2a)x^2 + (5a - 3)y^2 + (6 + ab)z^2$.
 39. $(m - mn + n)x^2 + (m - n + 4)y^2 + (1 - a)z^2$. 40. $(a - b)$
 $(x + y) + (a + b)(x - y) + (c + d)x$. 41. $m - n$. 42. $x - 1, x + 1$.
 43. $(125 - a)$ dollars. 44. $(6b - 25)$ cents. 45. $30 - x$.
 46. $x = \text{greater}, x - 5 = \text{less}$. 47. $(d - b - c)$ dollars.
 48. $(a - m - n - 100)$ dollars.

Page 38. — 49. $(50 - 25 - a - c)$ dollars. 50. $(3a - 40)$ dollars.
 51. $(b + d + e - f)$ dollars.

Page 39. — 2. 7. 3. 11. 4. 4. 5. 5. 6. 0. 7. 1. 8. 17.
 9. 8. 10. 16. 11. 4.

Page 40. — 13. 25. 14. 21. 15. 10. 16. 17. 17. 7. 19. 11 x .
 20. $11m - 4n$. 21. $6x^2 - 11ax$. 22. $3a^2 + 3b^2 + 2ab$. 23. $3xy - 5$.
 24. $c - 2b + 3$. 25. $7a - 5b$. 26. $3b - a - c$. 27. $5x$.
 28. $2x + 2y + 2z$. 29. $4a^2 - 5a + y + 1$. 30. $3n$. 31. $2x^2 + 2y^2$.
 32. $4x$. 33. $3x + 45$. 34. $a - 6$. 35. $2a + 3c$.

Page 44. — 2. $x = 6$. 3. $x = 7$. 4. $x = 2$. 5. $x = 3$. 6. $x = 3$.
 7. $x = 2$. 8. $x = 7$. 9. $x = 4$. 10. $x = 3$. 11. $x = 5$. 12. $x = 9$.
 13. $x = 5$. 14. $x = 3$. 15. $x = 2$. 16. $x = 1$. 17. $x = 2$. 18. $x = 2$.
 19. $x = 9$. 20. $x = 4$. 21. $x = 9$. 22. $x = 2$. 23. $x = 6$.
 24. $x = 6$. 25. $x = 4$. 26. $x = 7$. 27. $x = 9$. 28. $x = 5$. 29. $x = 12$.
 30. $x = 8$. 31. $x = 1$. 32. $x = 12$. 33. $x = 5$. 34. $x = 4$.
 35. $x = 15$. 36. $x = 4$. 37. $x = 2$. 38. $x = 8$. 39. $x = 2$.
 40. $x = 4$. 41. $x = 3$. 42. $x = 15$. 43. $x = 12$. 44. $x = 7$.
 45. $x = 6$. 46. $x = 5$. 47. $x = 4$. 48. $x = 5$. 49. $x = 9$.
 50. $x = 6$. 51. $x = 10$. 52. $x = 4$. 53. $x = 9$. 54. $x = 3$.
 55. $x = 4$. 56. $x = 10$. 57. $x = 3$.

Page 45. — 1. 4. 2. 12. 3. 2. 4. 14. 5. 5. 6. 37 cents,
 1st; 27 cents, 2d; 26 cents, 3d. 7. 3 cents, youngest; 7 cents,
 next; 10 cents, oldest. 8. 10 and 24. 9. \$2000, B; \$6000, A.
 10. \$12. 11. 5 miles, down hill; 18 miles, level; 9 miles, up hill.

Page 46. — 12. \$85, poorer horse; \$115, better horse. 13. 100
 gallons, first pipe; 200 gallons, 2d pipe; 300 gallons, 3d pipe.
 14. 325 cattle. 15. \$5, chain; \$55, watch. 16. 18 women; 22
 men; 50 children. 17. \$42,800, cargo; \$77,200, steamer. 18. \$600,
 clerk; \$1800, brother. 19. \$7500, house; \$2500 lot.

Page 50. — 3. 40. 4. $-40a$. 5. $-18a$. 6. $-110x$. 7. $108x$.
 8. $-42b$. 9. $51c^2$. 10. $-216x^2$. 11. $10x^4y^2z^2$. 12. $45x^2y^4$.
 13. $-64c^2d^2z^2$. 14. $126m^3n^3$. 15. $-125a^3b^3$. 16. $4(a + b)$.
 17. $-20(x - y)$. 18. $6ax + 4ay - 10az$. 19. $6x^3y - 6xy^2z$.
 20. $-12a^2x^2y + 20x^3y^2 - 8xy^3$. 21. $a^3b + 2a^2b^2 + ab^3$. 22. $-18m^2n$
 $-21m^2n^2 - 15mn^3$. 23. $9x^5 - 18x^5 + 45x^4 + 9x^3 - 27x^2$. 24. $108a^4b^2$
 $-156a^3b^3 + 48a^2b^4 - 72a^2b^2$. 25. $2ax^3 - 6ax^2y - 6ax^2z + 2axy^2z^2$.
 26. $12ab^3 + 9a^2b^3 - 15ab^4 - 6a^2b^2$. 27. $12m^3n - 40m^2n^2 - 32mn^3$.
 28. $-5x^5 - 5x^4 - 5x^3 - 5x^2 - 5x$. 29. $18a^4b^2 - 54a^3b^3 + 45a^2b^2c^2$
 $-60a^3b^3c + 42a^2b^2$.

Page 51. — 4. $x^2 + 2xy + y^2$. 5. $m^2 - n^2$. 6. $a^3 + 3a^2b$
 $+ 3ab^2 + b^3$. 7. $4a^2 - 25b^2$. 8. $x^2 - 6x - 40$. 9. $6y^2 + 13yz + 6z^2$.
 10. $9a^2 - 49b^2$. 11. $6x^2 - 9x - 6$. 12. $6a^2 + ab - 15b^2$.

13. $6m^2 + 17mn + 12n^2$. 14. $20y^2 - 32yz + 12z^2$. 15. $6b^2 + bc - 40c^2$.
 16. $24x^2 - 148x - 80$. 17. $20m^2 - 16mn + 25my - 20ny$. 18. $x^2 + 7xy + 10y^2$. 19. $1 - x^3$. 20. $x^2 - y^2 - 2y - 1$. 21. $6x^2 + 8xy - 8y^2$. 22. $2a^2 + ab - 4ac + 2bc - b^2$. 23. $12x^2 + 13x - 14$.
 24. $24a^2cm + 48ac^2 + 3ac^2m + 6c^3$. 25. $12x^2y^2 - 48y^2$. 26. $x^2 + 2xy + y^2 - xz - yz$. 27. $a^2 - 2ac + c^2 - ab + bc$. 28. $2a^2 - ax - ay + xy - x^2$. 29. $2x^2 + 13xy - 6x - 30y + 15y^2$.

- Page 52. — 30. $24x^2z^2 + 4xy^2z - 8y^4$. 31. $m^2 + 2m - n^2 + 1$.
 32. $25x^2 - 25x - 36$. 33. $9x^4 - 33x^2y^2 + 28y^4$. 34. $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$. 35. $8x^3 - 36x^2y + 54xy^2 - 27y^3$. 36. $m^5 - n^5$.
 1. 2. 2. 4. 3. 40 yr., father; 10 yr., son. 4. 25 ¢, Samuel; 15 ¢, John. 5. \$1500, A; \$500, B. 6. 10 yr. hence. 7. \$12.

- Page 53. — 8. 2 or 50. 9. $8x^3y^2z^2$. 10. df cents. 11. $(ab + 6a)$ miles. 12. $(4a - 8)$ animals. 13. $(20a - 3d)$ dollars. 14. $12a$ days. 15. $2b$ dollars, A; $1\frac{1}{2}c$ dollars, B. 16. $20m$ cents. 17. $(12b - ab)$ dollars. 18. $(600 - 40c)$ gallons. 19. $10ac$ dollars.

- Page 54. — 1. $x^2 + 2xy + y^2$. 2. $b^2 + 2bc + c^2$. 3. $m^2 + 4m + 4$.
 4. $a^2 + 2ax + x^2$. 5. $x^2 + 6x + 9$. 6. $y^2 + 2y + 1$. 7. $4x^2 + 4xy + y^2$. 8. $m^2 + 4mn + 4n^2$. 9. $9a^2 + 6ab + b^2$. 10. $4x^2 + 12xy + 9y^2$. 11. $a^2 + 8ab + 16b^2$. 12. $4m^2 + 8mn + 4n^2$. 13. $4x^2 + 20x + 25$. 14. $9m^2 + 6m + 1$. 15. $4a^2 + 20ab + 25b^2$. 16. $a^4 + 2a^2b^2 + b^4$. 17. $x^4 + 6x^2 + 9$. 18. $4m^4 + 12m^2n^2 + 9n^4$.
 19. $a^2b^2 + 4abc + 4c^2$. 20. $4x^2y^2 + 4xyz + z^2$. 21. $y^4 + 8y^2z^2 + 16z^4$. 22. $x^4 + 16x^2 + 64$. 23. $25a^2 + 70ab + 49b^2$. 24. $16a^4 + 24a^2b^2 + 9b^4$.

- Page 55. — 1. $a^2 - 2ax + x^2$. 2. $b^2 - 2bc + c^2$. 3. $m^2 - 2mn + n^2$. 4. $x^2 - 4x + 4$. 5. $y^2 - 2yz + z^2$. 6. $a^2 - 6ab + 9b^2$.
 7. $b^2 - 4bc + 4c^2$. 8. $4x^2 - 8xy + 4y^2$. 9. $b^2 - 10b + 25$. 10. $y^2 - 2y + 1$. 11. $a^2b^2 - 4ab + 4$. 12. $x^2 - 8x + 16$. 13. $4a^2 - 12ab + 9b^2$. 14. $m^2 - 4mn + 4n^2$. 15. $4b^2 - 16bd + 16d^2$.
 16. $a^4 - 4a^2b^2 + 4b^4$. 17. $b^2c^2 - 2bcxy + x^2y^2$. 18. $4x^4 - 20x^2y^2 + 25y^4$. 19. $4a^2 - 4ac + c^2$. 20. $9m^4 - 6m^2 + 1$. 21. $9m^2n^2 - 24mn + 16$. 22. $y^4 - 12y^2 + 36$. 23. $16x^4 - 40x^2y^2 + 25y^4$.
 24. $a^2b^2 - 4abc^2 + 4c^4$.

- Page 56. — 1. $a^2 - b^2$. 2. $m^2 - n^2$. 3. $a^2 - x^2$. 4. $4a^2 - b^2$.
 5. $4x^2 - y^2$. 6. $a^2 - 16$. 7. $4m^2 - 9n^2$. 8. $y^2 - 1$. 9. $x^2 - 25$.
 10. $4 - y^2$. 11. $a^2b^2 - 9c^2$. 12. $4m^2 - 4n^2$. 13. $b^2 - 4c^2$.
 14. $9x^2 - 64y^2$. 15. $x^2 - 100$. 16. $b^2c^2 - e^2f^2$. 17. $9x^4 - 4y^4$.
 18. $25a^2 - 9x^2$. 19. $a^4 - b^4$. 20. $m^2n^2 - 16$. 21. $x^2 - 36$.
 22. $16y^2 - 49z^2$. 23. $9x^2 - 16y^2$. 24. $4a^2b^2 - 25c^2$.

- Page 57. — 1. $x^2 + 7x + 12$. 2. $x^2 + 4x - 5$. 3. $x^2 - 5x + 6$.
 4. $x^2 - x - 42$. 5. $x^2 + 15x + 50$. 6. $x^2 - 10x - 39$. 7. $x^2 + 25x + 100$. 8. $x^2 + 4x - 32$. 9. $x^2 + 12x + 27$. 10. $x^2 - 6x - 72$.
 11. $x^2 - 12x + 35$. 12. $x^2 + 10x - 56$. 13. $x^2 + 7x - 8$.
 14. $x^2 - 9x + 20$.

- Page 58. — 15. $x^2 + 9x - 22$. 16. $x^2 - 29x + 100$. 17. $x^2 + 20x + 75$. 18. $x^2 - 9x + 18$. 19. $x^2 + 3x - 18$. 20. $x^2 + 21x + 108$.
 21. $x^2 + 2x - 120$. 22. $x^2 + 2xy - 8y^2$. 23. $x^2 - 8ax + 7a^2$.
 24. $x^2 + 16xy + 60y^2$.

Page 60. — 2. $x = 4; y = 2$. 3. $x = 6; y = 1$. 4. $x = 5, y = 4$.
 5. $x = 1; y = 3$. 6. $x = 3; y = 7$. 7. $x = 5; y = 7$. 8. $x = 9; y = 7$.
 9. $x = 1; y = 5$. 10. $x = 6; y = 1$. 11. $x = 8; y = 2$.
 12. $x = 5; y = 13$. 13. $x = 8; y = 9\frac{1}{2}$. 14. $x = 2; y = 2$. 15. $x = 3; y = 1$.
 16. $y = 12; z = 2$. 17. $y = 4; z = 5$. 18. $x = 5; z = 3$.
 19. $x = 2; y = 14$. 20. $x = 15; y = 15$. 21. $x = 4; y = 2$. 22. $y = 3; z = 13$.
 23. $x = 2; y = -2$. 24. $x = 3; y = 2$. 25. $x = 6; y = 1$.

Page 61. — 1. $x = 6; y = 4$. 2. $x = 8; y = 6$. 3. \$100, A; \$200, B. 4. 10 yr., son; 40 yr., father. 5. 10 marbles in one; 15 in other.
 6. 60 A. at \$30; 40 A. at \$40. 7. 9 oranges; 4 pears. 8. \$3000, A; \$2000, B.
 9. 5 and 7. 10. \$100, horse; \$35, cow.

Page 62. — 11. \$1 $\frac{1}{2}$, man; \$1, wife. 12. 20 and 6. 13. 6 one-dollar bills; 12 two-dollar bills.
 14. \$2, velvet; \$3, broadcloth. 15. \$13, A; \$11, B. 16. 5 cents, pad; 3 cents, pencil. 17. \$2, men; \$1, boys.
 18. \$1 $\frac{1}{2}$, sawing; \$ $\frac{1}{2}$, splitting.

Page 65. — 3. 2. 4. 8. 5. — 2. 6. — 5 y . 7. 5 ab . 8. — 3 mn
 9. 2 abx . 10. — 3 y^2z . 11. — 3 x^2 . 12. 3 c .

Page 66. — 13. 5 x^2 . 14. — x^2y^2 . 15. 2 x^2y^2 . 16. — 4 x . 17. 4 x^3 .
 18. — 5 cd . 19. — 3 xz^2 . 20. 8 m^3 . 21. 65 xz^2 . 22. 5 xz . 23. $ax - 2xy$.
 24. 3 $x + 5z^2$. 25. 2 $a^3b^2 + 7a$. 26. $z^2 - xz + 3$. 27. 2 $c - 7d$.
 28. 2 $xy - x - 3y$. 29. 4 $mn - 3m - 2n$. 30. 5 $bxy - 3ax^2 - 7cy^2$.
 31. $x - 4yz^2 + 5az^4$. 32. 6 $x^3 - 2x^2 + 4x + 5$. 33. 3 $abc + 4c - 2bc^2$.
 34. 29 $m^4 + 20m^2 - 28m - 23$. 35. 5 $x^2yz^2 - z - 2xz^2 - 6x^3y^2$.
 36. $x - 2x^3 + 3x^5 - 4x^7$. 37. 2 $x^2y^3 + 4xy^2 - 3y^4 + 5$. 38. 3 $x - 2y + 4ax^2 - 1$.
 39. $x^4 + x^3z - y^3z + x^2y^2z^2 - yz^3$. 40. 10 $x^2z^2 + 7xy - 3axz^2 - 4byz^3$.
 41. 2 $nx^2 - 3xy - 4my^3 + 3nx$. 42. 1 — $b(x + y) + ab^2(x + y)^2$.

Page 68. — 6. $x + y$. 7. $m + n$. 8. $x^2 + 2xy + y^2$. 9. $x + 3$.
 10. $x + 5$. 11. $a - 12b$.

Page 69. — 12. $x - 2$. 13. $x^2 - 7$. 14. $a^3 + 4$. 15. 5 $x - 3$. 16. $x^2 + x + 3$.
 17. 4 $a - 3b$. 18. $a^2 - ab + b^2$. 19. $x^2 + 2x + 4$. 20. $a^3 + y^3$. 21. $x - 2$.
 22. $x^2 + 10x + 14$. 23. 2 $x - 4y$. 24. 17 $x - 1$. 25. $a^2 - 2ax + 4x^2$.
 26. 9 $x^2 - 6xy + 4y^2$. 27. 2 $x^3 - 3x^2 + 2x$. 28. $x^3 - 3x - y$.
 29. $x^2 - 5x + 6$. 30. 1 — 3 $x + 2x^2 - x^3$. 1. $\frac{x + y}{2}$.
 2. $\frac{10}{x}$ cents. 3. $\frac{m}{2}$ apples. 4. $\frac{b}{a}$. 5. $\frac{2c}{5}$ cents.

Page 70. — 6. $\frac{15}{a}$ days. 7. 3 a animals. 8. $\frac{d}{c + 1}$, James;
 $\frac{cd}{c + 1}$, Henry. 9. $\frac{m + n}{b}$ dollars. 10. $\frac{2a + 2ab}{c}$ dollars.

11. $\frac{a + b + c}{3}$ cents. 1. 5 $(x + y) + 9z - 4$. 2. 3 $a^2 - 2b^2 - 5ab$.
 3. 2 $x + 2y + 2z$. 4. 3 $x^m + 5xy^n + 4y^n - 4$. 5. 1 $\frac{1}{3}$ $(m - n) - \frac{4}{5}(m + n)$.
 6. 7 $x + 11x^2 + 9x^3$. 7. $\frac{1}{4}a - \frac{3}{4}b + c$. 8. 8 $x^m y - 9x^2y^2 + 10x^m z^n$.

Page 71. — 9. $x^2 - 4z^2 - (x + y)^2 + 8$. 10. — $a^2x - 2m - 14$.
 11. $m^5 + 1$. 12. $a^2 + 2ab + b^2 + 2ac + 2ae + 2bc + c^2 + 2be + 2ce + e^2$.

13. $x^5 - 32y^5$. 14. $x^3 - 76x - 240$. 15. $a^4 - 2a^2b^2 + b^4$. 16. $16a^4 - 80a^3 + 500a - 625$. 17. $2a^2 + 3ab + b^2$. 18. $a^4 - a^3 + a^2 - a + 1$. 19. $m - c + z$. 20. $1 - 3x + 2x^2 - x^3$. 21. $2x^3 + 4x^2 + 8x + 16$. 22. $x = 2$. 23. $x = 4$. 24. $x = 5$. 25. $x = 7$. 26. $x = 10$. 27. $x = 3$. 28. $x = 9$; $y = 2$. 29. $x = 2$; $y = 4$. 30. $x = 8$; $y = 3$. 31. $x = 4$; $y = 4$.

- Page 72. — 32. \$250. 33. 40 mi., 1st day; 55 mi., 2d; 60 mi., 3d. 34. 7 days. 35. 10 days at \$1; 10 days at \$1½. 36. \$60, father; \$40, David. 37. \$25, overcoat; \$30, clothes; \$5, boots. 38. \$3. 39. 33½ pounds at 5 cents; 66½ pounds at 8 cents.

- Page 74. — 2. $9x(2x - 3y)$. 3. $5x^3y(3y + 4x)$. 4. $12mn^2(m - 4n)$. 5. $5ac(b - c + 3b^2)$. 6. $2x(3x + 2y - 4x^2)$. 7. $3x^2y(3y - 2 + 4z)$. 8. $3a^2b(2a^2 + 7 - 6ab^2)$. 9. $ab(3a - c - d)$. 10. $6a^3b^2(ab^3 + 6 - 7b)$. 11. $5a^2b^2(3b^2 + 4ab^3 - 5a^3)$. 12. $10m^2(2m - 5 + 3m^2)$. 13. $xy^2(1 + 9x + 27x^2y)$. 14. $5m(am + 2n + 3n^2)$. 15. $5x^2yz(9y + 12z)$. 16. $13x^2y(3y^2 - 5x + 7y^2)$. 17. $8a^3b^5(4 + 12a^3b^2 - a^5b^3)$. 18. $5a(5n + 15m^2 - 3n^2)$. 19. $5x^3(3a - 7y^3 - 11b)$. 20. $15x^4(3b^2 + 4y - 2z)$. 21. $36a^2y^3(2ay - 1 - 3y^2)$. 2. $(x + y)(a + b)$. 3. $(b + 3)(a + 2)$. 4. $(x + 3)(y + 3)$. 5. $(x + y)(a - b)$. 6. $(xy + 6)(1 - b)$. 7. $(y^2 + 1)(y - 1)$. 8. $(x^2 - y^2)(x + y)$. 9. $(x^2 - b^2)(y^2 + b)$. 10. $(a^3 + 2)(2a - 1)$. 11. $(x - b)(a - b)$. 12. $(x^2 + a)(x + 1)$. 13. $(a - b)(x^2 + y^2)$. 14. $(1 - a^3)(1 + a^2)$. 15. $(1 - x^2)(1 - x)$. 16. $x(x - y)(y - z)$. 17. $a(1 + y)(1 + y^2)$.

- Page 75. — 1. $(x + y)(x + y)$. 2. $(x + 2)(x + 2)$. 3. $(2a - b)(2a - b)$. 4. $(3m + n)(3m + n)$. 5. $(x - 2y)(x - 2y)$. 6. $(y + 1)(y + 1)$. 7. $(ab - 4)(ab - 4)$. 8. $(2n - 5)(2n - 5)$. 9. $(2x + 2y)(2x + 2y)$. 10. $(x - 5)(x - 5)$. 11. $(1 + z)(1 + z)$. 12. $(a^2 + 2b^2)(a^2 + 2b^2)$. 13. $(x - 3z)(x - 3z)$. 14. $(x + 10)(x + 10)$. 15. $(m + 4n)(m + 4n)$. 16. $(3x - 3y)(3x - 3y)$.

- Page 76. — 17. $(3 + a^2)(3 + a^2)$. 18. $(2a^2x - b^2)(2a^2x - b^2)$. 19. $(am - 4n)(am - 4n)$. 20. $(5x^2 + 8)(5x^2 + 8)$. 21. $(a^2 + 2x^2)(a^2 + 2x^2)$. 22. $(2bc - 3d)(2bc - 3d)$. 23. $(a^2 + 9)(a^2 + 9)$. 24. $(10x - 1)(10x - 1)$. 25. $(mn + 2)(mn + 2)$. 26. $(x^n + y^n)(x^n + y^n)$. 27. $(4 + 2m^2)(4 + 2m^2)$. 28. $(1 - 4x^m)(1 - 4x^m)$. 1. $(x + y)(x - y)$. 2. $(x^2 + 2)(x^2 - 2)$. 3. $(a + 3b)(a - 3b)$. 4. $(m + 1)(m - 1)$. 5. $(ab + c)(ab - c)$. 6. $(x + 5)(x - 5)$. 7. $(11x + 10y)(11x - 10y)$. 8. $(4a + 3b)(4a - 3b)$. 9. $(x^2 + 12)(x^2 - 12)$. 10. $(1 + z)(1 - z)$.

- Page 77. — 11. $(x^2 + y^2)(x + y)(x - y)$. 12. $(x + 4)(x - 4)$. 13. $(3x + 6)(3x - 6)$. 14. $(2x + 5)(2x - 5)$. 15. $(x^2 + 9)(x + 3)(x - 3)$. 16. $(mn + 8x)(mn - 8x)$. 17. $(2a + 4b)(2a - 4b)$. 18. $(x^2 + 25)(x + 5)(x - 5)$. 19. $(3x + 9)(3x - 9)$. 20. $(5ab + 7c^2d^2)(5ab - 7c^2d^2)$. 21. $(m^2 + 25)(m + 5)(m - 5)$. 22. $(15x + 10y)(15x - 10y)$. 23. $(4y^2 + 16)(2y + 4)(2y - 4)$. 24. $(ac + 3d)(ac - 3d)$. 25. $(x + 11)(x - 11)$. 26. $(cd + 1)(cd - 1)$. 27. $(5m + 15n)(5m - 15n)$. 28. $(x^3 + 1)(x^3 - 1)$. 29. $(x^m + y^n)(x^m - y^n)$. 30. $(x^{4m} + 13)(x^{4m} - 13)$.

Page 78.—3. $(a+4)(a+2)$. 4. $(x+7)(x-2)$. 5. $(x-7)(x+1)$. 6. $(m-9)(m-1)$. 7. $(b+3)(b+2)$. 8. $(n-2)(n+1)$. 9. $(n+2)(n-1)$. 10. $(x-9)(x+2)$. 11. $(x+6)(x-2)$. 12. $(a-8)(a+3)$. 13. $(x+9)(x-4)$. 14. $(y-2z)(y-z)$. 15. $(m-8n)(m+2n)$. 16. $(a+8x)(a+x)$. 17. $(x-20)(x+10)$. 18. $(a-11x)(a+5x)$. 19. $(x-4)(x-10)$. 20. $(x+5y)(x-4y)$. 21. $(m+6)(m+4)$. 22. $(a-7b)(a+3b)$. 23. $(x-8)(x-6)$. 24. $(x+9)(x-2)$. 25. $(c+21)(c-20)$. 26. $(a+7b)(a+4b)$. 27. $(x^n-5)(x^n-4)$. 28. $(x^2-14)(x^2+11)$. 29. $(m+17)(m-15)$. 30. $(x^6+7)(x^6+5)$.

Page 79.—4. $(x+y)(x^2-xy+y^2)$. 5. $(a+b)(a^2-ab+b^2)$. 6. $(m-n)(m^2+mn+n^2)$. 7. $(a-b)(a^2+ab+b^2)$. 8. $(m+n)(m^2-mn+n^2)$. 9. $(x-y)(x^2+xy+y^2)$. 10. $(s+t)(s^2-st+t^2)$. 11. $(c-d)(c^2+cd+d^2)$. 12. $(y+1)(y^2-y+1)$. 13. $(x-1)(x^2+x+1)$. 14. $(a-cd)(a^2+acd+c^2d^2)$. 15. $(m+s)(m^2-ms+s^2)$. 16. $(x-yz)(x^2+xyz+y^2z^2)$. 17. $(a+bc)(a^2-abc+b^2c^2)$. 18. $(mn+cd)(m^2n^2-cdmn+c^2d^2)$. 19. $(ax-by)(a^2x^2+abxy+b^2y^2)$. 20. $(x+1)(x^2-x+1)$. 21. $(y-1)(y^2+y+1)$.

Page 80.—3. $x=\pm 3$. 4. $x=\pm 5$. 5. $x=\pm 7$. 6. $x=\pm 2$. 7. $x=\pm 4$. 8. $x=\pm 6$. 9. $x=\pm 3$. 10. $x=\pm 4$. 11. $x=\pm 2$. 12. $x=\pm 2$. 13. $x=\pm 9$. 14. $x=\pm 6$. 15. $x=\pm 5$. 16. $x=\pm 10$. 17. $x=\pm 1$. 18. $x=\pm 4$. 19. $x=\pm 5a$. 20. $x=\pm 2c$.

Page 81.—23. $y=9$ or 1 . 24. $x=13$ or -5 . 25. $x=11$ or 5 . 26. $z=2$ or -14 . 27. $x=3$ or -17 . 28. $x=15$ or 5 . 29. $y=13$ or 5 . 30. $x=1$ or -23 . 31. $x=-1$ or -5 . 32. $x=3$ or -27 . 33. $x=5$ or -7 . 34. $y=10$ or 4 . 35. $z=10$ or -40 . 36. $x=1$ or -25 . 37. $x=10$ or -50 . 38. $x=27$ or -1 . 39. $y=6$ or -4 . 40. $x=15$ or -65 .

Page 82.—43. $x=-5$. 44. $x=-6$. 45. $x=-3$. 46. $x=9$. 47. $x=-8$. 48. $x=-13$. 49. $x=7$. 50. $x=-15$. 51. $x=14$. 52. $x=-10$. 53. $x=-12$. 54. $x=-18$. 55. $x=25$. 56. $x=-11$. 57. $x=50$. 58. $x=-22$. 59. $x=-30$. 60. $x=-40$.

Page 83.—63. $x=5$ or -3 . 64. $x=-5$ or -1 . 65. $x=-9$ or -1 . 66. $x=4$. 67. $x=-8$ or 6 . 68. $x=-8$ or -5 . 69. $x=8$ or -3 . 70. $x=-4$ or -3 . 71. $x=2$ or -11 . 72. $x=-5$ or -10 . 73. $x=2$ or -1 . 74. $x=7$ or 11 . 75. $x=10$ or -12 . 76. $x=25$ or -3 . 77. $x=4a$ or $2a$. 78. $x=6$ or -9 . 79. $x=12a$ or $-8a$. 80. $x=-8b$ or $-3b$.

Page 84.—83. $x=-1$ or -9 . 84. $x=2$ or -10 . 85. $x=13$ or 5 . 86. $x=1$ or -5 . 87. $x=15$ or 5 . 88. $x=1$ or -15 . 89. $x=-2$ or -20 . 90. $x=4$ or -8 . 91. $y=2$ or -18 . 92. $x=17$ or 7 . 93. $x=24$ or 6 . 94. $x=5$ or -45 . 95. $x=14$ or -6 . 96. $x=1$ or -13 . 97. $x=22$ or 16 . 98. $y=1$ or -3 . 99. $x=8$ or -2 . 100. $x=2$ or -10 .

Page 86.—3. $7xyz$. 4. $9xyz$. 5. $5ab$. 6. $11m^2n^2z$. 7. $20abx^2$. 8. $9bm^2n$. 9. $x-1$. 10. $x+y$. 11. $m+n$.

Page 87.—12. $a-b$. 13. $x+1$. 14. $x-2$. 15. $y-1$. 16. $1+a$. 17. $x+3$. 18. $x+3$. 19. $x-4$. 20. x^2-1 . 21. $x-1$. 22. $a+b$. 23. $x+7$. 24. $x-2y$. 25. $x-5$. 26. $x-4$. 27. $x-2y$.

28. $x + 7y$. 29. $x - y$. 30. $x - 5$. 31. $x - 2y$. 32. $x - 3$.
 33. $a + 2x$. 34. $b + c$. 35. $x - 1$. 36. $a + 3$. 37. $a - 2y$.
 38. $2x + 6y$.

- Page 89. — 3. $72 a^3 b^2 c^2$. 4. $30 a^2 b^2 x^2$. 5. $70 a^2 b^2 c^2 x^2 y$.
 6. $891 a^2 m^2 y^2$. 7. $105 x^3 y^3 z^2$. 8. $x^3 - 2x^2 - 4x + 8$. 9. $x^3 + x^2 y - xy^2 - y^3$. 10. $a^2(x^2 - y^2)$. 11. $a^4 - b^4$.
 Page 90. — 12. $x^3 - 11x^2 - 4x + 44$. 13. $x^3 + 9x^2 + 26x + 24$.
 14. $x^5 + 2x^4 - 16x - 32$. 15. $c^3 - 9c^2 + 24c - 16$. 16. $m^2x(a^2 - b^2)$.
 17. $x^2(a^3 - b^3)$. 18. $16a^4 - 8a^3 + 2a - 1$. 19. $a^3 - 4a^2 - 17a + 60$.
 20. $xy(x^2 - y^2)$. 21. $abxy(x^3 - 1)$. 22. $x^4 - a^4$. 23. $a^3 - 6a^2b + 9ab^2 - 4b^3$.
 24. $x^3 + 5x^2 - 36x - 180$. 25. $x^4 - x^3 + x - 1$.
 26. $x^3 + 3x^2 - x - 3$. 27. $ab^2m(a^3 + a^2b - ab^2 - b^3)$. 28. $105ab^2(a^2 - b^2)$.
 29. $x^3 + x^2 - 14x - 24$. 30. $x^3 - 7x^2 + 7x + 15$.
 31. $x^4 - 2x^3y - 52x^2y^2 + 98xy^3 + 147y^4$. 32. $x^3 - 6x^2 - 19x + 84$.
 33. $x^3 - 4x^2 - 47x + 210$. 34. $x^4 + x^3y - xy^3 - y^4$. 35. $x^3 - 13x - 12$.
 36. $x^8 - 1$.

- Page 93. — 2. $\frac{18x}{30}$. 3. $\frac{15mn}{24}$. 4. $\frac{16a^2b}{28}$. 5. $\frac{24ac^2d^2xy}{45b^2c^3d^2}$.
 6. $\frac{6ax - 15bx}{33x}$. 7. $\frac{12m - 9n^2}{48}$. 8. $\frac{4x^2y}{6ax + 4x}$. 9. $\frac{12abc}{10cx - 6cy}$.
 10. $\frac{16axy - 20bxy}{12xy}$.

- Page 94. — 11. $\frac{7a - 7b}{a^2 - b^2}$. 12. $\frac{(m+n)^2}{m^2 - n^2}$. 13. $\frac{30a - 35}{75}$.
 14. $\frac{m+n}{(m+n)^2}$. 15. $\frac{2ax - 4a}{x^2 - 4}$. 16. $\frac{x^2 - y^2}{(x+y)^2}$. 2. $\frac{c}{3b}$. 3. $\frac{y}{3xz}$.
 4. $\frac{2}{7acx}$. 5. $\frac{1}{3n}$. 6. $\frac{4x}{9y}$. 7. $\frac{xy}{3z^3}$. 8. $\frac{m^2}{4nx}$. 9. $\frac{z^2}{5a^2x^2}$.
 10. $\frac{1}{2x - 3a}$.

- Page 95. — 11. $\frac{3b}{a - b^2}$. 12. $\frac{1}{a - b}$. 13. $\frac{1}{m + n}$. 14. $\frac{1}{a - b}$.
 15. $\frac{x - y}{x + y}$. 16. $\frac{1}{x + 1}$. 17. $\frac{1}{x + 1}$. 18. $\frac{a + 3}{a - 7}$. 19. $\frac{m + 5}{m - 5}$.
 20. $\frac{x + 4}{x - 6}$. 21. $\frac{x - 4}{x - 1}$. 22. $\frac{1}{m^2 + mn + n^2}$. 23. $\frac{x + 4}{x + 7}$.
 24. $\frac{x + y}{x^2 + xy + y^2}$. 25. $\frac{x - 1}{x^2 - x + 1}$. 26. $\frac{1}{9(x - y)}$.
 27. $\frac{1}{4x^2 + 6xy + 9y^2}$. 28. $\frac{x - 1}{2y}$.

- Page 96. — 2. $\frac{10x + 3y}{2}$. 3. $\frac{20x - 2y}{5}$. 4. $\frac{32x + y}{4}$.
 5. $\frac{20x - 6y}{10}$. 6. $\frac{9x - 1}{7}$. 7. $\frac{5x - 4}{3}$. 8. $\frac{40x + 5y}{6}$.
 9. $\frac{40a - 4z - x}{5}$. 10. $\frac{11m - n}{3}$. 11. $\frac{a - b + cx}{c}$. 12. $\frac{18x - y - z}{9}$.

13. $\frac{6b+3c}{2}$. 14. $\frac{9x-5y}{7}$. 15. $\frac{3ax^2+6a-x}{ax}$.
 16. $\frac{a^2-2ac+c^2}{a}$. 17. $\frac{1+2x}{1+x}$. 18. $\frac{3x^2}{x^2-1}$. 19. $\frac{4ab-a}{b}$.
 20. $\frac{2a^2}{a-x}$. 21. $\frac{a^2+2ac-2c^2}{a-c}$. 22. $\frac{x^2+x-8}{x+2}$. 23. $\frac{m^2-4n^2+14}{m+2n}$.
 24. $\frac{m^2+2mn+n^2-5}{m+n+x}$. 25. $\frac{2a^2-5}{a+x}$. 26. $\frac{am^2}{m-n}$.

Page 97. — 2. $2a + \frac{5a}{9}$.

3. $2ab + \frac{4ab}{11}$.

4. $3xy^2z$.

5. $4ac + \frac{4c}{9}$.

6. $a + \frac{c^2}{a}$.

7. $2x - \frac{5y}{6x}$.

8. $2ax - x^3$.

9. $a+x$.

10. $x^2+x+\frac{x+1}{x-1}$.

11. $2a-2b-\frac{2b^2}{a+b}$.

12. $5y+x+\frac{x}{a}$.

13. $2b-\frac{b^2}{a+b}$.

14. $2x+8+\frac{39}{x-4}$.

15. $x^2+xy+y^2+\frac{2y^3}{x-y}$.

16. $2+b-\frac{a}{b}$.

17. $x+3+\frac{2x+4}{x^2-x-1}$.

Page 98. — 18. $2a+2b$.

19. $x-1$.

20. $2x+6+\frac{23}{x-3}$.

21. $a+b+\frac{2b^2}{a-b}$.

22. $x-2+\frac{2}{x-2}$.

23. $x-1+\frac{1-x}{2x-3}$.

24. $x-1+\frac{4-x}{5x-1}$.

25. $x^2-x+1-\frac{2}{x+1}$.

Page 99. — 2. $\frac{ac}{bc}, \frac{b^2}{bc}$.

3. $\frac{mx}{nx}, \frac{abn}{nx}$.

4. $\frac{9bxy^2}{3bcy}, \frac{2ac}{3bcy}$.

5. $\frac{3aby}{12y}, \frac{4cd}{12y}$.

6. $\frac{2bz}{x^2z^2}, \frac{3cdx}{x^2z^2}$.

7. $\frac{a^2x^2}{axy}, \frac{bcy}{axy}, \frac{abx}{axy}$.

8. $\frac{axyz}{xyz}$.

$\frac{bcz}{xyz}, \frac{acy}{xyz}, \frac{abx}{xyz}$.

9. $\frac{2az}{axyz}, \frac{2yz}{axyz}, \frac{2x}{axyz}$.

Page 100. — 10. $\frac{bc^2m}{a^2b^2c^2}, \frac{a^2bcn}{a^2b^2c^2}, \frac{a^2mn}{a^2b^2c^2}$.

11. $\frac{10axyz}{x^2y^2z}, \frac{8by^2z}{x^2y^2z}$.

$\frac{9x}{x^2y^2z}$.

12. $\frac{12x}{78}, \frac{52x}{78}, \frac{3xy}{78}, \frac{8y}{78}$.

13. $\frac{2ay+2by}{6xy}, \frac{3ax-3bx}{6xy}$.

$\frac{abxy}{6xy}$.

14. $\frac{acx+acy}{4ac}, \frac{2ax-2ay}{4ac}, \frac{2cx^2+2cy^2}{4ac}$.

15. $\frac{5x-5y}{x^2-y^2}, \frac{6xy}{x^2-y^2}$.

$\frac{7x+7y}{x^2-y^2}, \frac{x+y}{x^2-y^2}$.

16. $\frac{3a-3b}{a^2-b^2}, \frac{4a+4b}{a^2-b^2}$.

17. $\frac{3x^2-6x}{x(x^2-4)}, \frac{a-b}{a^3-b^3}$.

$\frac{5x^2+10x}{x(x^2-4)}, \frac{9x^2-36}{x(x^2-4)}$.

18. $\frac{a+b}{a^3-b^3}, \frac{4x}{4(x^2-1)}$.

$\frac{a^2+ab+b^2}{a^3-b^3}, \frac{a-b}{a^3-b^3}$.

19. $\frac{4x-4}{4(x^2-1)}, \frac{3x-3}{4(x^2-1)}$.

$\frac{4x}{4(x^2-1)}$.

20. $\frac{2a^2+2b^2}{a^4-b^4}$.

$$\frac{a^3 + a^2b + ab^2 + b^3}{a^4 - b^4}, \quad \frac{3a^2 - 3b^2}{a^4 - b^4}, \quad 21. \frac{x^2 + 4x + 4}{x^3 - x^2 - 4x + 4}, \quad \frac{x^2 - 4x + 4}{x^3 - x^2 - 4x + 4}.$$

$$22. \frac{x^2 - 5x}{x^3 - 3x^2 - 13x + 15}, \quad \frac{ax - a}{x^3 - 3x^2 - 13x + 15}, \quad 23. \frac{4x + 4}{4x(x - 2)},$$

$$\frac{x^2 - x}{4x(x - 2)}, \quad \frac{2x - 4}{4x(x - 2)}, \quad 24. \frac{5 - 10x}{1 - 4x^2}, \quad \frac{3x + 6x^2}{1 - 4x^2}, \quad \frac{4 - 13x}{1 - 4x^2},$$

$$25. \frac{a^3 - a^2b + ab^2 - b^3}{a^4 - b^4}, \quad \frac{a^3 + ab^2}{a^4 - b^4}, \quad \frac{3a}{a^4 - b^4}, \quad 26. \frac{x^2y - y^3}{y(x^3 + y^3)},$$

$$\frac{xy + y^2}{y(x^3 + y^3)}, \quad \frac{x^4 - x^3y + x^2y^2}{y(x^3 + y^3)}.$$

Page 103. — 4. $x = 8$. 5. $x = 15$. 6. $x = 4$. 7. $x = 16$.
8. $x = 12$. 9. $x = 3$. 10. $x = 6$. 11. $x = 12$. 12. $x = 14$. 13. $x = 9$.
14. $x = 30$. 15. $x = 24$.

Page 104. — 16. $x = 3$. 17. $x = 10$. 18. $x = 24$. 19. $x = 35$.
20. $x = 15$. 21. $x = 10$. 22. $x = 4$. 23. $x = 44$. 24. $x = 5$.
25. $x = 6$. 26. $x = 12$. 27. $x = 22\frac{5}{6}$. 28. $x = 4$. 29. $x = 1\frac{1}{3}$.
30. $x = 2$. 31. $x = 8$. 32. $x = 13$. 33. $x = 5$. 34. $x = \frac{3}{7}$.
35. $x = 3\frac{5}{16}$. 36. $x = 41$. 37. $x = 17$. 38. $x = 7$.

Page 105. — 39. $x = 1$. 40. $x = 9$. 41. $x = \frac{3}{4}$. 42. $x = 2\frac{3}{11}$.
43. $x = 40$. 44. $x = 16$. 45. $x = 3$. 2. 12. 3. \$7200. 4. 75 gal-
lons. 5. 16 years. 6. 15 and 20.

Page 106. — 7. \$75, horse; \$50, wagon. 8. 20 and 56. 9. 420.
10. 10 and 40. 11. 50 years, father; 20 years, son. 12. 100 sheep.
13. 10 children, 90 cents. 14. \$18, father; \$12, oldest son; \$7,
youngest son. 15. \$4225. 16. \$135, A; \$140, B; \$165, C.

Page 108. — 4. $\frac{5}{ab}$. 5. $\frac{2ab + 5ad}{3xy}$. 6. $\frac{nx + my}{ny}$.

7. $\frac{2x^2 + 3y}{6ax}$. 8. $\frac{2a^2}{a^2 - x^2}$. 9. $\frac{5a^2 - b^2}{4a - 4b}$. 10. $\frac{13a}{12c}$. 11. $\frac{4x^2 - 2xy}{x^2 - y^2}$.

12. $\frac{2a^2 + 14a + 26}{a^2 + 8a + 15}$. 13. $\frac{6x^2 - 9xy + 4y^2}{2x^2 - 3xy + y^2}$. 14. $\frac{12m + 2n}{m^2 - n^2}$.

15. $\frac{2x^2 + 2}{x^2 - 1}$. 16. $\frac{y^2 - 2xy}{x^2 - xy}$. 17. $\frac{3x^2 + 3y^2}{x^3 - y^3}$. 18. $\frac{2a^2 - 6a + 43}{a^2 - 3a - 10}$.

19. $\frac{2a^2 + 2b^2}{a^2 - b^2}$. 20. $\frac{12x - 2}{4x^2 - 1}$. 21. $\frac{1}{x - y}$. 22. $\frac{a^2 + x^2}{a^2x - ax^2}$.

23. $\frac{3a^2 - ax}{a^2 - x^2}$. 24. $\frac{2x}{9}$. 25. $\frac{2a - b}{5}$. 26. $\frac{abcx - mny}{axy}$.

27. $\frac{cdx - 2bz}{xyz}$.

Page 109. — 28. $\frac{1}{x^2 - 3x + 2}$. 29. $\frac{3x + 13}{x^2 - 4}$.

30. $\frac{4ab}{a^3 - a^2b - ab^2 + b^3}$. 31. $\frac{x^2 - x - 9}{x^2 + 5x + 6}$. 32. $\frac{3x - 11}{x^2 - 2x}$.

33. $\frac{-4ax}{x^4 - 2a^2x^2 + a^4}$. 34. $\frac{5c+2d}{6}$ dols. 35. $\frac{3b}{5}$ dols. paid out, $\frac{2b}{5}$ left.
 36. $\frac{3a+3b}{2}$ dols. 37. $\frac{53xy}{105}$. 38. $\frac{3}{4}x$. 39. $\frac{19}{66}ab$. 40. $\frac{5m-5n}{6}$.
 41. $\frac{23x}{20}$. 42. $\frac{39a-28b}{63}$. 43. $\frac{37ax}{36}$. 44. $\frac{5a+7b}{60}$. 45. $\frac{x+y}{y}$.
 46. $\frac{x+y-2xy^2}{x^2-y^2}$. 47. $\frac{2}{1-a}$. 48. $\frac{2a-2c}{ac}$. 49. $\frac{4x+8}{x^2-1}$.

- Page 110. — 50. $\frac{6x-22}{x^2-x-2}$. 51. $\frac{6x^2+4x-6}{x^3-7x-6}$. 52. $\frac{2x}{x^4-5x^2+4}$.
 53. $\frac{2x^2}{x^2-1}$. 54. $\frac{2a^2}{x^3-a^2x}$. 55. $\frac{4a^4-4a^3x+4a^2x^2}{a^4-x^4}$. 56. $\frac{7a}{4}$ miles.
 57. $\frac{7a+7c}{5}$ dollars. 58. $\frac{2e+2f}{3}$ miles.

- Page 112. — 3. $\frac{m^2n^2}{cx}$. 4. $\frac{10c^2d}{9x^2}$. 5. $\frac{3d}{4b}$. 6. $\frac{2bm}{5x^2yz}$.
 7. $\frac{7m^2}{6bx}$. 8. $\frac{x+y}{a^2x}$. 9. $\frac{m-n}{3(m+n)}$. 10. $\frac{11m^2n^2}{x-y}$. 11. $\frac{10a^2b}{x^2-y^2}$.
 12. $\frac{3b}{x+2}$. 13. $\frac{a^2}{x^2(a+x)}$. 14. $\frac{1}{5(x+1)}$. 15. $\frac{a(a-3b)}{a^2-b^2}$.
 16. $\frac{x-5}{x+3}$. 17. $\frac{3}{m^2-1}$. 18. $\frac{x+3}{5(x-7)}$. 19. $\frac{4}{x-8}$. 20. $\frac{b(a^2-b^2)}{a}$.
 21. $\frac{x+6}{6}$.

- Page 113. — 22. $\frac{1}{a}$. 23. a . 24. $\frac{4-x}{2}$. 25. $\frac{x}{x-y}$.
 26. $(x+2)$. 27. $\frac{y}{2}$. 28. $\frac{a}{3y^2z^2}$. 29. $\frac{1}{m^2}$. 30. 1. 31. $\frac{bc}{a}$ dollars.
 32. $\frac{6c}{b}$ pounds. 33. $\frac{n(3a+4)}{3}$ dollars. 34. $\frac{2d(d-e)}{3}$ dollars.
 35. $\frac{c}{b}, \frac{5c}{b}, \frac{ac}{b}, \frac{c^2+cd}{b}$ miles.

- Page 115. — 3. $\frac{nx}{my}$. 4. $\frac{ac}{ry}$. 5. xz . 6. $\frac{3ab^2}{2x^3}$. 7. $\frac{8a^3}{3dxy}$.
 8. $\frac{25x^3y^5z}{4a^3b^4c}$. 9. $\frac{1}{x+y}$. 10. $\frac{1}{(a+b)x}$. 11. $\frac{a+2x}{a^2+4x^2}$. 12. $\frac{x}{y(x-y)}$.
 13. $\frac{dxy}{2}$. 14. $\frac{2}{x-2}$. 15. $\frac{2}{x^2+xy+y^2}$. 16. $\frac{1}{5}$. 17. $\frac{xy(b+1)}{b}$.
 18. $\frac{ax}{m(x-1)}$. 19. $\frac{5b(x-10)}{3ax}$. 20. $\frac{a+x}{a-x}$. 21. $\frac{2(x-6)}{3x^2}$.
 22. $\frac{x^2-x-2}{x^2-9x+20}$. 23. $a-1$. 24. $\frac{x+y}{x-y}$. 25. $4(x-y)^2$.

- Page 116.** — 26. $\frac{x+y}{m^2-mn+n^2}$. 27. $\frac{x-1}{x+3}$. 28. $\frac{ab}{7}$ dollars.
 29. $\frac{2c}{a}$. 30. $\frac{4m-n}{c}$ dollars. 31. $\frac{af}{d}$ cows. 32. $\frac{2(a-b)}{3m}$.

- Page 117.** — 2. $x=1$; $y=6$. 3. $x=2$; $y=10$. 4. $x=7$; $y=2$.
 5. $x=4$; $y=1$. 6. $x=12$; $y=15$. 7. $x=18$; $y=14$. 8. $x=7$;
 $y=7$. 9. $x=5$; $y=11$.

- Page 118.** — 10. $x=7$; $y=2$. 11. $x=6$; $y=5$. 12. $x=8$;
 $y=11$. 13. $x=7$; $y=9$. 14. $x=6$; $y=15$. 15. $x=4$; $y=9$.
 16. $x=18$; $y=12$. 17. $x = \frac{cm-bn}{ac-ab}$, or $\frac{bn-cr}{ab-ac}$; $y = \frac{m-n}{b-c}$, or
 $\frac{n-m}{c-b}$. 18. $x=10$; $y=15$. 19. $x = \frac{2b-ac}{4-a^2}$; $y = \frac{2c-ab}{4-a^2}$.
 20. $x=9$; $y=8$. 21. $x=5$; $y=10$.

- Page 119.** — 2. $x=3$; $y=7$. 3. $x=6$; $y=1$. 4. $x=5$; $y=6$.
 5. $x=10$; $y=9$. 6. $x=2$; $y=8$. 7. $x=2$; $y=17$. 8. $x=19$;
 $y=13$. 9. $x=12$; $y=4$. 10. $x=3$; $y=4$. 11. $x=14$; $y=10$.

- Page 120.** — 12. $x=16$; $y=12$. 13. $x=15$; $y=18$. 14. $x=5$;
 $y=7$. 15. $x=25$; $y=40$. 16. $x=8$; $y=16$. 17. $x=9$; $y=4$.
 18. $x = \frac{1}{2}a$; $y = \frac{1}{2}b$. 19. $x=15$; $y=20$. 20. $x = \frac{ad-bc}{dm-bn}$;
 $y = \frac{ad-bc}{an-cm}$.

- Page 121.** — 1. $x=3$. 2. $x=6$. 3. $x=4\frac{1}{2}$. 4. $x=3$. 5. $x=a+b$.
 6. $x = \frac{ab+b^2}{a+c-1}$. 7. $x = \frac{a+b}{c^2-2}$. 8. $x = \frac{a+b}{a-b}$. 9. $x=3$.
 10. $x=40$. 11. $x=12$. 12. $x=14$. 13. $x=12$. 14. $x=2$.
 15. $x=10$. 16. $x=4$.

- Page 122.** — 17. $x=14$. 18. $x = \frac{ac+abd}{ad+d}$. 19. $x = \frac{b}{a-1}$.
 20. $x=1$. 21. $x=4$. 22. $x=22$. 23. $x=-7$. 24. $x = 1\frac{1}{2}$.
 25. $x=1\frac{1}{2}$. 26. $x=8$. 27. $x=4$. 28. $x=2$.

- Page 123.** — 29. $x=4\frac{1}{2}$. 30. $x=2$. 31. $x=3\frac{3}{8}$. 32. $x=3$.
 33. $x=4$. 34. $x=-1\frac{9}{21}$. 35. $x=-2$. 36. $x=3$. 37. $x=6$.
 38. $x=1$. 39. $x=\frac{7}{17}$. 40. $x=3$. 41. $x=a+b$. 42. $x = \frac{1}{ab}$.

- Page 124.** — 43. $x=b-1$. 44. $x=1$. 45. $x = \frac{c}{d}$. 46. $x=2$;
 $y=1$. 47. $x=5$; $y=7$. 48. $x=6$; $y=-4$. 49. $x=15$; $y=10$.
 50. $x=6$; $y=16$. 51. $x=2$; $y=2$. 52. $x=2$; $y=2$. 53. $x = \frac{a+b}{2}$;
 $y = \frac{a-b}{2}$. 54. $x = \frac{ac+bd}{a^2+b^2}$; $y = \frac{bc-ad}{a^2+b^2}$. 55. $x = \frac{2a}{m+n}$;
 $y = \frac{2b}{m-n}$. 56. $x=mn$; $y = \frac{m}{n}$. 57. $x = \frac{ab}{a+b}$; $y = \frac{ab}{a+b}$.

Page 125.—1. 60 and 40. 2. A and B, 9 days; C, 18 days.
3. House, \$3500; invested, \$2500. 4. 72 apples. 5. 20 yards,
longer; 8 yards, shorter. 6. 50 rods, length; 30 rods, breadth.
7. \$800. 8. 500 soldiers.

Page 126.—9. A, 120 acres; B, 180 acres; C, 150 acres.
10. 40 apples. 11. 24 rods, length; 12 rods, breadth. 12. 24.
13. \$400. 14. 30 and 40. 15. 100 eggs. 16. 10 and 14.

Page 127.—17. A, \$300; B, \$500. 18. \$1000. 19. 10 days.
20. \$1000. 21. 720 apples. 22. 60 cents. 23. Horse, \$100;
wagon, \$50. 24. 16 acres.

Page 128.—2. $x = 2, y = 1, z = 3$. 3. $x = 4, y = 1, z = 5$.

Page 129.—4. $x = 5, y = 3, z = 5$. 5. $x = 4, y = 10, z = 14$.
6. $x = 6, y = 1, z = 8$. 7. $x = 3, y = 7, z = 11$. 8. $x = 12, y = 14$,
 $z = 20$. 9. $x = 18, y = 10, z = 9$. 10. $x = 5, y = 6, z = 7$.
11. $x = 2, y = 12, z = 5$. 12. $x = 11, y = 13, z = 4$. 13. $x = 10$,
 $y = 4, z = 9$. 14. $x = 4, y = 4, z = 8$. 15. $x = 12, y = 16, z = 20$.
16. $x = 8, y = 10, z = 3$. 17. $x = 3, y = 6, z = 9$. 18. $x = 20$,
 $y = 10, z = 5$. 19. $x = 8, y = 5, z = 2$.

Page 130.—20. $x = 2, y = 3, z = 6$. 21. $x = 5, y = 5, z = 10$.
22. $x = 10, y = 14, z = 13$. 23. $x = 20, y = 25, z = 30$. 24. $x = 3$,
 $y = 5, z = 7$. 25. 30 sheep, 1st; 40 sheep, 2d; 60 sheep, 3d.
26. 10 cents, Henry; 15 cents, James; 25 cents, Ralph. 27. 150 bu.
wheat; 50 bu. oats; 40 bu. barley.

Page 132.—3. $4x^4y^2$. 4. $16x^2y^2z^4$. 5. $9x^6z^2$. 6. $8a^6b^6$.
7. $-64a^3b^6x^3$. 8. $9c^4d^4$. 9. $-x^9y^3z^3$. 10. $125x^{12}y^6$. 11. $81a^8b^4c^8$.
12. $16m^8n^4$. 13. $-1024a^5m^{15}$. 14. $32a^5b^{10}d^5$. 15. $-216a^6m^3$.
16. $256x^{32}y^8$. 17. $-128a^{21}b^{14}$. 18. $-100000a^5z^{15}b^5$. 19. $a^{18}b^{18}c^{45}d^9$.
20. $a^{24}b^{32}m^8$. 21. $-128a^{35}b^{14}d^7e^7$. 22. $-x^{36}y^{27}z^{63}$. 23. $x^{10}y^5m^5z^{5n}$.

Page 133.—24. $64x^{24}y^{24}z^{48}$. 25. $-128b^{14}c^{7m}d^{14n}$. 26. $a^{24}b^{16}x^{24}z$.
27. $625a^8b^4y^{12}z^{16}$. 28. $-a^{12}y^{15n}z^{9n}$. 29. $2^na^{2n}b^nc^nx^ny^n$. 30. $\frac{4a^2b^2}{9m^2x^2}$.

31. $\frac{27x^6y^3}{64a^6z^6}$. 32. $\frac{x^8y^4z^{12}}{16a^4m^{12}n^4}$. 33. $\frac{a^{10}b^{10}c^{5n}}{x^{5m}y^{10}z^5}$. 34. $\frac{125a^6x^3y^3}{216m^3n^6z^3}$.
35. $\frac{a^{2n}b^{2n}c^{8n}}{x^{4n}y^{4n}z^{2n}}$. 36. $\frac{729a^{18}x^{24}y^6}{4096m^{12}n^6z^{24}}$. 37. $\frac{a^{24}x^{6n}}{y^{32}z^{40}}$. 38. $\frac{4096a^8b^4}{2401x^8y^4}$.
39. $\frac{a^{9n}x^{12n}}{b^{12n}y^{9n}}$. 40. $-\frac{x^{28}y^7}{10000000a^{14}m^{21}}$. 41. $\frac{b^{18}c^9d^{36}}{m^{18}n^{27}x^{45}}$. 1. $x^2 - 2xy + y^2$.

2. $9a^2 + 12ab + 4b^2$. 3. $25a^2 + 40ac + 16c^2$. 4. $x^3 + 3x^2y$
 $+ 3xy^2 + y^3$. 5. $x^3 - 3x^2y + 3xy^2 - y^3$. 6. $a^4 + 4a^3b + 6a^2b^2$
 $+ 4ab^3 + b^4$. 7. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$. 8. $m^2 - 6mn + 9n^2$.
9. $x^2 + 4xy + 4y^2$. 10. $4a^2 - 20ab + 25b^2$. 11. $27x^3 + 54x^2y$
 $+ 36xy^2 + 8y^3$. 12. $64x^3 - 144x^2y + 108xy^2 - 27y^3$. 13. $a^5 + 5a^4b$
 $+ 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$. 14. $x^3 + 3x^2 + 3x + 1$. 15. a^4
 $+ 2a^2b^2 + b^4$. 16. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$. 17. $a^2 + b^2$
 $+ c^2 + 2ab - 2ac - 2bc$. 18. $x^2 + y^2 + z^2 - 2xy - 2xz + 2yz$.
19. $4x^2 + 4y^2 + z^2 + 8xy - 4xz - 4yz$. 20. $a^{2n} + b^{2n} + c^{2n} + 2a^nb^n$
 $+ 2a^nc^n + 2b^nc^n$. 21. $a^{2m} + b^{2m} + c^{4m} - 2a^mb^m - 2a^mc^{2m} + 2b^mc^{2m}$.

- Page 136.** — 2. $\pm 2ab$. 3. $2xy^2$. 4. $\pm 5x^2y^2$. 5. $-3mn$.
 6. $\pm 2ab^2c^5$. 7. $-a^2b^5c^3$. 8. $-5x^3yz^5$. 9. $\pm a^2b^4cd^3$. 10. $\pm 7m^4n^5$.
 11. $\pm 3a^3b^4e$. 12. $-2a^2x^3y^4$. 13. $x^2y^5z^4$. 14. $-4x^ny^3z^{4n}$.
 15. $\pm 10a^2bc^5$. 16. $-7m^2x^3z^4$. 17. $\pm x^2y^5z^3$. 18. $\pm a^nb^4c^{2n}$.
 19. $-2m^2n^3x^5y^8$. 20. $\pm \frac{2ab^2}{3x^2y^3}$. 21. $\pm \frac{5m^2n^4}{6a^5b^3}$. 22. $\frac{2ab^3c^5}{3x^2y^4z^7}$.

- Page 137.** — 23. $-\frac{3a^2b^3}{7x^5y^3}$. 24. $\frac{x^2y^3z^5}{a^4b^5c^7}$. 25. $\pm \frac{9x^5y^6}{12m^8n^{10}}$.
 26. $\pm \frac{2a^2b^3c^8}{4x^4y^6z^2}$.

- Page 139.** — 3. $x+2y$. 4. $a-3$. 5. $4x+3y$. 6. $3x-2y$.
 7. $5m+6$. 8. $10a-1$. 9. $9a+5$. 10. $6x+8y$. 11. $a+b-c$.
 12. $x+y+4$. 13. $x-2y-2z$. 14. $2a-3b+2$. 15. $2a+b-1$.
 16. a^2-a-2 . 17. $2x^2+x+1$. 18. $3a-c+5$. 19. $4m^2+4m-4$.
 20. $5a^2+3a+3$. 21. m^2+3m+1 . 22. $x-y+2$. 23. $m^2+3m-2n$.
 24. $2a^2-b+6$. 25. $x^2-2x+\frac{1}{2}$. 26. $x+y-z$.

- Page 142.** — 3. 27. 4. 81. 5. 98. 6. 73. 7. 85. 8. 66.
 9. 54. 10. 49. 11. 95. 12. 97. 13. 235. 14. 163. 15. 118.
 16. 321. 17. 425. 18. 503. 19. 431. 20. 515. 21. 989. 22. 664.
 23. 4.38. 24. 24.9. 25. .0795. 26. 3.55. 27. 4.16. 28. 18.22.
 29. 3.71.

- Page 144.** — 3. $a+b$. 4. $x-y$. 5. $x+2$. 6. $3a+1$.

- Page 145.** — 7. $3x-y$. 8. x^2+4 . 9. $2m-n$. 10. x^2+x .
 11. $am+2b$. 12. $2a^2-5c^2$. 13. $3+2b$. 14. $1+a$. 15. m^2+6 .
 16. $2x+3y$. 17. $4x^3+12y^2$. 18. x^2-ax-a^2 . 19. $2x-7$.
 20. $7x+4y$. 21. a^3+3b^3 . 22. $2x^2+6y^2$. 23. a^2b-3 . 24. $1+3x$.
 25. m^4-n . 26. x^2-2x+1 . 27. a^2-3a+2 . 28. x^2-x-1 .
 29. $3m^2-5m-2$. 30. $2x^2+3xy-y^2$. 31. $1-3x+4x^2$.
 32. $1+4x+4x^2$.

- Page 149.** — 4. 83. 5. 53. 6. 64. 7. 63. 8. 89. 9. 97. 10. 68.
 11. 85. 12. 255. 13. 354. 14. 465. 15. 177. 16. 327. 17. 345.
 18. 409. 19. 906. 20. 25. 21. 352. 22. 237. 23. 159. 24. 655.
 25. 3.04. 26. 50.7. 27. 3.64.

- Page 151.** — 3. $x=\pm 4$. 4. $x=\pm 3$. 5. $x=\pm 5$. 6. $x=\pm 6$.
 7. $x=\pm 2$. 8. $x=\pm 3$. 9. $x=\pm 1$. 10. $x=\pm 14$. 11. $x=\pm 2$.
 12. $x=\pm 5$. 13. $x=\pm 2$. 14. $x=\pm 8$. 15. $x=\pm 3$. 16. $x=\pm 9$.
 17. $x=\pm 10$. 18. $x=\pm(a-3)$. 19. $x=\pm 2a$. 20. $x=\pm a$. 1. 2.
 2. 6 and 15. 3. 7. 4. 17 rods. 5. 12 yards.

- Page 152.** — 6. 5. 7. 10 rods, length; 6 rods, breadth.
 8. 26 years.

- Page 154.** — 4. 2 or -6. 5. 5 or -3. 6. 3 or -11. 7. 4 or -10.
 8. 7 or -1. 9. 1 or -11. 10. 10 or -2. 11. 5 or -25.
 12. 2 or -4. 13. 2 or -7. 14. 2 or -4 $\frac{1}{2}$. 15. 5 or 3 $\frac{1}{2}$. 16. 6 or 1.
 17. 4 or -14. 18. 1 or -13. 19. 32 or -2. 20. 2 or -16.
 21. 12 or -14. 22. 8 or 1. 23. 30 or -2. 24. 5 or -16. 25. 12 or 3.
 26. 9 or -11. 27. 4 or -1.

Page 155.—3. 5 or $-3\frac{2}{3}$. 4. 3 or $-3\frac{1}{2}$. 5. 1 or $-1\frac{3}{4}$. 6. 4 or $\frac{1}{2}$.
 7. 5 or $-5\frac{1}{2}$. 8. 7 or -4. 9. 8 or $-7\frac{1}{2}$. 10. 1 or $-\frac{1}{4}$. 11. 2 or $-1\frac{2}{3}$.
 12. 1 or -18. 13. 5 or 4. 14. 3 b or -b. 15. 2 a or $-5a$.
 16. 4 or $-3\frac{1}{6}$. 17. 5 or $-4\frac{2}{3}$. 18. 13 or $-4\frac{1}{3}$. 19. 6 or $-1\frac{1}{2}$.
 20. 6 or -2. 21. $\frac{a+4}{a}$ or -1. 22. $\frac{2a+b}{2a}$ or -1. 23. 5 or -6.
 24. 10 or -8.

Page 156.—25. $2c$ or $\frac{c}{2}$. 26. $a+m$ or $a-m$. 27. $2a$
 or $\frac{1}{2}a$. 2. 9 and 4. 3. 10 and 16 or -16 and -10. 4. 2 and 6 or
 -6 and -2. 5. 52 rows; 54 trees in a row. 6. 20 rods; 25 rods.
 7. 3 or $-2\frac{1}{2}$.

Page 157.—8. 11 and 15 or -15 and -11. 9. 8 or $-6\frac{1}{3}$.
 10. 8 men. 11. 10. 12. \$16. 13. 10 yards. 14. 4 cattle.
 15. \$100. 16. \$30.

Page 159.—5. $x=2$; $y=1$. 6. $x=13$; $y=7$. 7. $x=1$
 or $\frac{1}{4}$; $y=\frac{1}{4}$ or 1. 8. $x=4$; $y=4$. 9. $x=5\frac{1}{3}$ or 2; $y=\frac{1}{8}$ or 7.
 10. $x=\pm 4$; $y=1$ or 9. 11. $x=8$; $y=5$. 12. $x=6\frac{7}{9}$ or 5;
 $y=-1\frac{2}{9}$ or 3. 13. $x=7$; $y=4$. 14. $x=5$ or 2 or -2 or -5; $y=2$
 or 5 or -5 or -2. 15. $x=18\frac{1}{2}$; $y=15\frac{1}{2}$. 16. $x=10$ or 2; $y=2$ or 10.

Page 160.—17. $x=5$ or -4; $y=4$ or -5. 18. $x=6$; $y=4$.
 19. $x=5$ or $-9\frac{4}{5}$; $y=2$ or $11\frac{1}{5}$. 20. $x=5$ or -1; $y=1$ or -5.
 21. $x=10$ or 9 or -9 or -10; $y=9$ or 10 or -10 or -9. 22. $x=11$
 or -5; $y=14$ or -2. 23. $x=2$ or $-1\frac{1}{5}$; $y=5$ or -9. 24. $x=1$ or 5;
 $y=4$ or 2. 25. $x=4$ or 3; $y=-3$ or -4. 26. $x=8$ or 1; $y=-1$ or -8.
 27. $x=\frac{a+2b}{2}$ or $\frac{a-2b}{2}$; $y=\frac{a-2b}{2}$ or $\frac{a+2b}{2}$. 28. $x=\frac{5a}{2}$; $y=\frac{b}{2}$.
 1. 9 and 7. 2. 8 and 3. 3. 24 years, A; 20 years, B. 4. 16 in.,
 length; 12 in., breadth. 5. 15 rods; 10 rods.

Page 161.—6. 22 and 12. 7. 5 and 2. 8. 150 miles, whole distance;
 90 miles, A; 60 miles, B. 9. 40 yards, linen; 60 yards, cotton.
 10. 45 rods, length; 30 rods, breadth. 11. 10 and 8. 12. 120 yards
 silk; 80 yards velvet. 13. 20 rods, 15 rods, rectangle; 10 rods, square;
 or 16 rods, 21 rods, rectangle; 8 rods, square.

Page 162.—1. $2\frac{3}{17}$. 2. $(7a+2b+5)(x-y)$. 3. $2x+30$.
 4. $18x^2-16x-7\sqrt{x}$. 5. $10y^2-6ax(a-1)-4(b^2-2)$. 6. $\frac{ab}{4}$.
 7. $(600-4b-7c-5bc)$ dollars. 8. $170x^2-70y^2+90c^2-15$.
 9. $2x^2-2y^2+2z$. 10. $a+1$. 11. x^5-32y^5 . 12. x^6-26x^4
 + $169x^2-144$.

Page 163.—13. $5xy(x-2a+5y^2-3a^2xy)$. 14. $(x+13)(x-12)$;
 $(x-8)(x-7)$; $(x-10)(x+7)$. 15. $\frac{ab-b}{7}$ cents. 16. $a-10c=2$.
 17. x^2-xy+y^2 . 18. $(a-b^2)(a^2+ab^2+b^4)$; $(5x-4y)(25x^2$
 + $20xy+16y^2)$; $(x^2+9)(x+3)(x-3)$. 19. $2x^3-6x^2+3x-1$.
 20. $\frac{ab-1}{bc+d}$. 21. $(2a+9)(2a+9)$; $25(2x+y)(2x-y)$;

- $(3x - 7y)(3x - 7y)$. 22. $\frac{6b + 5c + ab}{11 + b}$ cents. 23. $a + b$.
 24. $8(x + 4)(x^2 - 4x + 16)$; $(ab + 1)(ab - 1)$; $(m + 3)(m^2 - 3m + 9)$.
 25. $x = 7$. 26. $x^2 + 4xy + 4y^2$; $16a^2 - 24ab + 9b^2$; $4m^2 - 20m + 25$;
 $9a^2b^2 + 6ab + 1$; $25 + 20a + 4a^2$; $36x^2 - 84xy + 49y^2$; $a^4 + 20a^2 + 100$;
 $m^4 - 8m^2n^2 + 16n^4$; $x^4 - 6x^2y + 9y^2$; $9a^2 + 30ax + 25x^2$.
 Page 164.—27. $x^2 + 3x - 28$; $x^2 - 225$; $x^2 + 13x + 30$; $x^2 - 3x - 108$;
 $4x^2 - 25$; $m^4 - 1$; $x^2 - 19x + 88$; $x^2 + x - 210$; $9x^2 - 49$. 28. $a - 2$. 29. $x + 3$.
 30. $x^3 - 7x + 6$. 31. $x^3 + 6x^2 - 9x - 54$. 32. $\frac{1}{a - b}$. 33. $\frac{x - 2y}{x - 5y}$.
 34. $\frac{2}{x - y}$. 35. $\frac{x - 3}{3}$. 36. $\frac{1}{2(y - 1)}$. 37. $\frac{x^2}{a^2 - x^2}$.
 38. $\frac{(a - b)(a - x)}{x}$. 39. $\frac{85a - 20b}{84}$.

- Page 165.—40. $\frac{1}{m}$, A; $\frac{1}{n}$, B; $\frac{m + n}{mn}$, both; $\frac{mn}{m + n}$, days. 41. $x = 10$.
 42. $x = 2$. 43. $x = 3$. 44. $x = a - b$. 45. $\frac{a}{10}$, horses; $\frac{3a}{10}$,
 cows; $\frac{3a}{5}$, sheep. 46. $x = \frac{ab}{a + b - c}$. 47. $x = \frac{2}{7}$. 48. $\frac{am}{m + n}$,
 $\frac{an}{m + n}$. 49. $\frac{15m}{m^2 - 1}$. 50. $\frac{10abc}{ab + ac + bc}$.

- Page 166.—51. $x = \frac{1}{2}$; $y = \frac{1}{3}$. 52. $x = 2\frac{1}{5}$; $y = 14\frac{1}{5}$. 53. $x = 7$;
 $y = 10$. 54. $x = \frac{13}{3b - 2a}$; $y = \frac{13}{5a - b}$. 55. $x = 2\frac{1}{2}$; $y = -\frac{5}{12}$;
 $z = 4\frac{1}{3}$. 56. $x = 24$; $y = 60$; $z = 120$. 57. $x = \frac{a + b - c}{2}$;
 $y = \frac{a - b + c}{2}$; $z = \frac{b - a + c}{2}$. 58. $x = \pm 5$. 59. $x = \pm \frac{1}{a - 1}$.
 60. $x = \pm 8$. 61. $x = 4$ or -1 . 62. $x = 2$ or -17 . 63. $x = 3$
 or $-2\frac{3}{4}$. 64. $x = a + m$ or $a - m$. 65. $x = \frac{2a + 5}{2a}$ or -1 .
 66. $x = 1$ or $-1\frac{1}{2}$.

- Page 167.—67. $x = 13$ or -8 . 68. $x = 12$ or 6 . 69. $x = 14$
 or -2 . 70. $x + 2y = 3$. 71. 245. 72. $x = 5$ or $-2\frac{5}{6}$; $y = 3$ or $6\frac{7}{6}$.
 73. $x = 6$ or 1 or -1 or -6 ; $y = 1$ or 6 or -6 or -1 . 74. $x = 10$;
 $y = 8$. 75. $x = \pm 7$; $y = 2$ or -12 . 76. \$96, Tom; \$84, John.
 77. 570 votes. 78. 60 cents. 79. 60 sheep.

- Page 168.—80. $\frac{4}{5}$. 81. \$1.50, wheat; \$.40, corn. 82. 45 feet,
 60 feet. 83. \$180, carriage; \$160, horse. 84. 15 and 19. 85. $262\frac{1}{2}$
 gallons. 86. 8 and 2 or $4\frac{1}{2}$ and $-1\frac{1}{2}$. 87. 15 men; 5 women.

- Page 169.—88. 50 sheep, A; 30 sheep, B. 89. 11 and 19.
 90. \$.35, tea; \$.30, coffee; \$.05, sugar. 91. 24 miles. 92. \$150.
 93. \$140, horse; \$30, cow; \$10, sheep. 94. 7 and 3 or -7 and -3 .
 95. \$60.

- Page 170.—96. \$312. 97. 162 sheep, 1st; 144 sheep, 2d; 128
 sheep, 3d. 98. \$3600. 99. 11 and 9. 100. $1\frac{1}{2}$ yards. 101. 117 mi.,

A; 130 mi., B. 102. 12 rods, equal lots; 25 rods, larger lot: or 1 rod, equal lots; 14 rods, larger lot. 103. 4 rods.

Page 171. — 104. $a + x$. 105. $3x + 1$. 106. $x + 1$. 107. $30xy(x^2 - y^2)$. 108. $x^3 + 18x^2 + 107x + 210$. 109. $12xy(x^2 - y^2)^2$.

110. $\frac{3(a+b)}{4ab(a-b)}$. 111. $\frac{a^2 - a + 1}{a^2 + 2a + 1}$. 112. $\frac{120x + 30}{15(x-2)}$, $\frac{10x - 5}{15(x-2)}$, $\frac{9x + 6}{15(x-2)}$. 113. $\frac{x^4 - 6x^3 + 5x^2 - 24x - 36}{x^4 + 6x^3 + 5x^2 - 24x - 36}$, $\frac{x^4 - 6x^3 + 5x^2 - 24x - 36}{x^4 - y^4}$, $\frac{x^4 - 6x^3 + 5x^2 - 24x - 36}{x^4 - y^4}$.

114. $\frac{x^4 - x^3y + x^2y^2 - xy^3}{x^4 - y^4}$, $\frac{x^3y + xy^3}{x^4 - y^4}$. 115. $\frac{12 - 12x}{4(1 - x^2)}$, $\frac{5 - 5x}{4(1 - x^2)}$, $\frac{8x}{4(1 - x^2)}$.

Page 172. — 116. $\frac{ax}{(a-x)(a+2x)}$. 117. $\frac{y}{x+y}$. 118. $x = 7$. 119. $x = 4$. 120. $x = -2\frac{1}{4}$. 121. $x = 2$. 122. $x = 1\frac{2}{3}$. 123. $x = bm$. 124. $x = 1$. 125. $x = \frac{ab-1}{bc+d}$. 126. $x = \frac{2}{abc}$. 127. $x = \frac{8}{5}a$. 128. $x = \frac{1}{2}$.

Page 173. — 129. $x = 3$; $y = 2$; $z = 1$. 130. $x = 2$; $y = 4$; $z = 6$. 131. $x = a$; $y = b$. 132. $x = \frac{b^2 + c^2 - a^2}{2a}$; $y = \frac{a^2 - b^2 + c^2}{2b}$.

133. $x = a - \frac{abc^2}{ac+b}$; $y = \frac{b^2}{ac+b}$. 134. $x = (a+b)^2$; $y = (a-b)^2$. 135. 94. 136. 75. 137. 327. 138. 3.24. 139. .0321. 140. .0671. 141. 2187. 142. 6561. 143. $2x^2 - x + 3$. 144. $4x^2 + 3x + 10$. 145. $x + y + 3z$. 146. $z + m - 1$. 147. $x^3 - 3x^2 + 4x - 5$.

Page 174. — 148. 709. 149. 805. 150. 46.8. 151. 9.36. 152. 20.53. 153. 1.111. 154. $2x^2 + 4ax - 3a^2$. 155. $3m^2 - 4m - 7$. 156. $a^3 - a^2 - a + 1$. 157. $x = 7$ or 1 or -1 or -7; $y = 1$ or 7 or -7 or -1. 158. $x = 8$ or -3; $y = 1\frac{1}{2}$ or -4. 159. $x = 3$ or -2; $y = 2$ or -3. 160. $x = \frac{a+b}{2}$; $y = \frac{b-a}{2}$. 161. $x = 6$; $y = 4$.

162. $x = \frac{1}{4}$; $y = \frac{1}{2}$. 163. $x = \frac{a}{2}$; $y = \frac{b}{2}$. 164. $x = 4$ or 3 or -3 or -4; $y = 3$ or 4 or -4 or -3. 165. 64 inches. 166. 96 feet.

Page 175. — 167. 4 P.M. 168. 80 miles. 169. 4 days. 170. \$9000. 171. \$670, D; \$605, C. 172. \$8. 173. 12 days. 174. $1\frac{1}{3}$ cords. 175. \$40, lost.

Page 176. — 176. 35 miles. 177. 92,160 acres. 178. 12 inches. 179. 8 feet; $4\frac{1}{2}$ pounds, weight per foot. 180. \$56, first horse; \$33, second horse. 181. $5a^4 + 4a^3 + 3a^2 + 2a + 1$. 182. $21\frac{3}{4}$ hours, equal faucets; $13\frac{7}{11}$ hours, the other. 183. 55 acres, at \$20 an acre; 45 acres, at \$30 an acre.

$$p^2 + x^2 = 10001x^2$$

